GENERIC VANISHING, GAUSSIAN MAPS, AND FOURIER-MUKAI TRANSFORM

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A quite basic fact about abelian varieties (and complex tori) is that the only line bundle in Pic^0 with non-trivial cohomology is the structure sheaf. Mumford, in his treatment of the dual abelian variety, made the far-reaching remark that this yields the following sheaf-theoretic formulation. Let A be an abelian variety (over an algebraically closed field) of dimension g, let us denote $\widehat{A} = \mathrm{Pic}^0 A$ and let $\mathcal{P}$ be a Poincaré line bundle on $A \times \widehat{A}$. Then ([M]§13, see also and [K], Th. 3.15 for the complex-analityc setting)

$$R^i p_{\widehat{A}*}\mathcal{P} = \begin{cases} 0 & \text{for } i < g \\ k(0) & \text{for } i = g \end{cases} \tag{0.1}$$

where $k(0)$ denotes the one-dimensional skyscraper sheaf at the identity point of $\widehat{A}$. It is worth to remark that the Fourier-Mukai equivalence between the derived categories of A and $\widehat{A}$ ([Mu1]) is a direct consequence of (0.1).

The theme of *generic vanishing* can be seen as a vast generalization of the above to *varieties mapping to abelian varieties (or complex tori).* Given a morphism $a : X \to A$ from a compact Kähler manifold to a complex torus (e.g. the Albanese map), works of Green-Lazarsfeld and Simpson ([GE1],[GE2],[S], see also [EL]) provide a fairly complete description of the loci

$$V^i_a = \{\xi \in \widehat{A} \mid h^i(X, a^*P_\xi) > 0\,\}$$

(here P_ξ denotes the line bundle parametrised by the point $\xi \in A^\vee$ via the choice of the Poincaré line bundle $\mathcal{P}$).

The main result of this paper is instead a generalization of (0.1), conjectured by Green and Lazarsfeld themselves. The methods of proof are completely algebraic and very different from the Hodge-theoretic ones of Green-Lazarsfeld. As a byproduct we get an algebraic proof of Green-Lazarsfeld's *Generic Vanishing Theorem* working on any algebraically closed field as well, under the separability assumption.

Theorem 1. (compare [GL2], Problem 6.2) *On an algebraically closed field, let $a : X \to A$ be a separable morphism from an irreducible, Gorenstein variety X to an abelian variety A. Then*

$$R^i p_{\widehat{A}*}((a, id_{\widehat{A}})^*\mathcal{P}) = 0 \quad \textit{for } i < \dim a(X).$$

Corollary 2. *On an algebraically closed field, let $a : X \to A$ be a separable morphism from an irreducible, Gorenstein variety X to an abelian variety A. Then:*

(a) (compare [GL1]) $\quad codim_{\widehat{A}} V^i_a \geq max\{0, \dim a(X) - i\}$.

Hence the V^i_a's are proper subvarieties of $\widehat{A}$ for $i < \dim a(X)$ (Generic Vanishing Theorem). *In particular, if the morphism a is generically finite then the varieties V^i_a are proper for $i < \dim X$ and, therefore, $\chi(\omega_X) \geq 0$.*

(b) compare ([EL], Lemma 1.8) $\;V^i_a \subset V^{i+1}_a$ *for any $i \leq \dim a(X)$.*

After completing the proof of Theorem 1 the author was informed that Christopher Hacon had already proved, by completely different methods, Green-Lazarsfeld's conjecture for smooth complex projective varieties ([H]).

Let us sketch the proof of the Corollary. (b) Follows by base change (e.g. [M],§5 Cor.2). (a) In the first place one can assume that the morphism a is generically finite (in the general case one reduces – using relative Serre vanishing – to a multiple hyperplane section Z of dimension equal to $\dim a(X)$, such that $a_{|Z}$ is generically finite and separable). Next, one considers the loci

$$V^i_a(\omega_X) = \{\xi \in \widehat{A} \mid h^i(\omega_X \otimes a^* P_\xi) > 0\}$$

By Serre duality we have that $V^i_a = -V^{\dim X - i}_a(\omega_X)$. Therefore (a) is equivalent (under the assumption that A is generically finite) to the inequality

$$codim_{\widehat{A}}(V^i_a(\omega_X)) \geq i. \tag{0.2}$$

For a generically finite morphism a Theorem 1 states that $R^i p_{\widehat{A}*}\mathcal{Q} = 0$ for $i \neq \dim X$ and this, by Grothendieck duality, yields that

$$\mathcal{E}xt^i_{\mathcal{O}_{\widehat{A}}}(R^{\dim X} p_{\widehat{A}*}\mathcal{Q}, \mathcal{O}_X) \cong \begin{cases} R^i p_{\widehat{A}*}(\mathcal{Q}^\vee \otimes p_X^* \omega_X) & \text{for } i \leq \dim X \\ 0 & \text{for } i > \dim X. \end{cases} \tag{0.3}$$

Therefore

$$codim_{\widehat{A}}(Supp(R^i p_{\widehat{A}*}(\mathcal{Q}^\vee \otimes p_X^* \omega_X))) \geq i. \tag{0.4}$$

Thus (0.2) follows by descending induction on i: the case $i = \dim X$ follows from (0.4) and base change. For $i < \dim(X)$, let W be a component of $V^i_a(\omega_X)$. If W is also a component of the support of $R^i p_{\widehat{A}*}(\mathcal{Q}^\vee \otimes p_X^* \omega_X)$ then (0.4) applies. Otherwise, by base change as in (b), $W \subset V^{i+1}_a(\omega_X)$ and (0.2) follows by induction.

As a disclaimer one should point out that in this paper we don't recover the main result of [GL2] (which is crucial for most applications, as the ones of [EL]) i.e. that the positive-dimensional components of the loci V^i_a are translates of subtori. This will be the object of further research by the author.

Let us now turn to a more detailed overview of the methods of proof. The other existing approaches ultimately reduce the problem, via some duality theory, to a question about sheaves of holomorphic forms: Green-Lazarsfeld, via Hodge duality, reduce the infinitesimal study of the loci V^i_a to facts concerning the groups $H^0(\Omega^j_X \otimes a^* P_\xi)$, while Hacon, via a Grothendieck duality argument in the derived category, reduces (0.2) to Kollär's results on vanishing and semi-simplicity properties of higher direct images of the canonical bundle. At the opposite, we tackle directly the sheaves $R^i p_{A^\vee_*}\mathcal{Q}$. The argument is divided in two independent steps, each of them – in the author's hope – of independent interest:

(i) a vanishing criterion (Theorem 3 below) for the higher direct images of relative line bundles, in terms of first order infinitesimal deformations of suitable one-dimensional multiple hyperplane sections. (As a particular case, one has a vanishing criterion for higher cohomology groups of line bundles which does not seem to be noted before.) This part is independent on Theorem 1 and works in a very general setting.
(ii) a verification, based on Fourier-Mukai theory, of the hypotheses of the above vanishing criterion for the line bundle $(a, id_{\widehat{A}})^*\mathcal{P}$ on $X \times \widehat{A}$ of Theorem 1 .

(i) First-order vanishing criterion. Let us describe the first step. We introduce certain linear maps, called *global co-gaussian maps.* They generalize in various directions the notion of gaussian

maps, introduced by J. Wahl ([W1],[W2]) and studied by Wahl himself and other authors in connection of the *extendibility problem* (see e.g. *loc cit* and [BM], [Z], [BeEL]).

Specifically, let X be a Cohen-Macaulay projective variety of dimension $n+1$. We consider a flag $C \subset Y$ where Y is a hyperplane section of X and C is the complete intersection of $n = \dim X - 1$ divisors linearly equivalent to X (if $\dim X = 2$ we agree that $C = Y$). Let

$$e_C^{X,Y} \in \mathrm{Ext}^1(\Omega^1_{Y|C}, N^{-1}_{|C}) \tag{0.5}$$

be the extension class of the sequence $0 \to N^{-1}_{|C} \to \Omega^1_{X|C} \to \Omega^1_{Y|C} \to 0$. Finally, let $(T, \mathcal{Q})$ be a *projective family of line bundles* on X, i.e. T is a projective scheme and $\mathcal{Q}$ a line bundle on $X \times T$. The global co-gaussian map associated to such data is a certain linear map $\phi_n^{\mathcal{P}_T}$ whose source is $\mathrm{Ext}^1(\Omega^1_{Y|C}, N^{-1}_{|C})$ (see §1 for the definition). The following result holds:

Theorem 3. (First-order vanishing criterion) *In the above setting, assume that:*
(a) $Y \sim kD$ with D ample divisor on X and k big enough.
(b) $\mathcal{R}^i p_{T_*}\mathcal{Q} = 0$ for any positive $i < n$.
Then: $\mathcal{R}^n p_{T_*}\mathcal{Q} = 0$ if and only if $e \in \ker(\phi_n^{\mathcal{P}_T})$.

The fact that we are dealing only with *(dim -1)*-cohomology is not restrictive since a standard argument with hyperplane sections and Serre vanishing always allows to reduce to this case. Moreover it should be said that, although Theorem 3 is true, we prove it under the additional assumption that $\mathcal{R}^0 p_{T_*}\mathcal{Q} = 0$, which allow a simpler proof (see §3 for the precise statement). The meaning of Theorem 3 relies on very classical remarks. We give a very rough outline of the matter, which takes §1-3:
- As in classical algebraic geometry, one sees a higher cohomology group (resp. a higher direct image) of a line bundle (resp. of a family of line bundles) as the defect of completeness of a linear series (resp. a relative linear series). E.g., in the setting of Theorem 3, $\mathcal{R}^n p_{T_*}\mathcal{Q}$ is the cokernel of the restriction map

$$\rho^{X,C} : p_{T_*}(p_X^*\mathcal{O}_X(nY) \otimes \mathcal{Q}) \to p_{T_*}(p_X^*\mathcal{O}_X(nY)_{|C} \otimes \mathcal{Q})$$

where: p_T and p_X are the projections of $X \times T$, Y is a Cartier divisor $\sim kD$ with D ample and k is big enough, $n = \dim X - 1 = \dim Y$, and C is the complete intersection of n divisors linearly equivalent to Y .
- The defect of completness of a linear series (resp. relative linear series) can be seen as the obstruction to lift a certain map to projective space (resp. to the projectivisation of a coherent sheaf). In the setting above the cokernel of $\rho^{X,C}$ is the obstruction to lift the relative projective map $f_T : C \times T \to \mathcal{P}_T$ to $X \times T$, where $\mathcal{P}_T$ is the projectivisation of the coherent sheaf $p_{T_*}(p_X^*\mathcal{O}_X(nY)_{|C} \otimes \mathcal{Q})$. (The reader will notice that, stricly speaking, this is not true unless $p_{T_*}\mathcal{Q} = 0$, as for the Poincarè line bundle. However we will neglect this point, since it disappears at first order).
- In extremely rough terms, one can somewhat differentiate (with respect to C) the above condition (the liftability of the projective map f_T), by replacing the ambient variety X with a sort of first-order embedded (in X) deformation $\widetilde{Y}_C$ associated to the flag $C \subset Y$. Roughly, this could be described as the "double structure induced on C by the divisor $2Y$ in X". The intrinsic (i.e. non-embedded) isomorphism class of $\widetilde{Y}_C$ is $e_C^{X,Y}$ of (0.5) and the fact that $e_C^{X,Y} \in \ker(\phi_n^{\mathcal{P}_T})$ means that the map f_T can be lifted to a first order embedded deformation which is isomorphic, as non-embedded deformation, to $\widetilde{Y}_C \times T$. The content of Theorem 3 is that, for k big enough, this is

equivalent to the fact that the map f_T can be lifted to $X \times T$. An elementary example of these ideas is shown at the beginning of §1.

(ii) Comparing certain first-order deformation spaces via Fourier-Mukai functor. The second step, which takes §4-7, consists in a cohomological computation showing that, given a generically finite and separable map a from a Gorenstein variety X to an abelian variety A, and taking as family of line bundles the line bundle $\mathcal{Q} = (a \times id_{\widehat{A}})^*\mathcal{P}$ on $X \times \widehat{A}$, then $e_C^{X,Y} \in \ker(\phi_n^{\mathcal{P}_{\hat{A}}})$ (where $\phi_n^{\mathcal{P}_{\hat{A}}}$ is the global co-gaussian map of the previous point). In view of Theorem 3, this proves Theorem 1 in the case when a is generically finite and this is enough since it is always possible, via suitable hyperplane sections, to reduce to that case. In fact we prove a stronger fact : given a generically finite map $a : Y \to A$ from a n-dimensional Gorenstein variety Y to an abelian variety A and a 1-dimensional multiple hyperplane section C of Y such that the map $a_{|C}$ is birational onto its image, we consider the natural map $\phi_n^A : \operatorname{Ext}^1(\Omega^1_{Y|C}.N^{-1}_{|C}) \to \operatorname{Ext}^1(a^*_{|C}\Omega^1_A, N^{-1}_{|C})$ obtained from the map a. It is quite clear that, if X is as above, and Y and C are recovered from X as in point (i), then $e_C^{X,Y} \in \ker(\phi_n^A)$. The result is

Theorem 4. $\ker(\phi_n^A) \subset \ker(\phi_n^{\mathcal{P}_{\hat{A}}})$. (Here, for simplicity, we are assuming that the curve C can be chosen to be smooth. Otherwise there is a slightly different definition of the map ϕ_n^A, see Theorem 4.5 for the precise statement).

As suggested by the notation, the map ϕ_n^A, associated to the the map $a : Y \to A$, is somewhat analogous to the map $\phi_n^{\mathcal{P}_{\hat{A}}}$ associated to the $\widehat{A}$-projective map $C \times \widehat{A} \to \mathcal{P}_{\widehat{A}}$ described in point (i) (although this analogy is not perfect unless $n = 1$, and this is the reason of some of the technical complications of this paper). So Theorem 4 can be seen as a geometric consequence at the first-order deformation level of the duality between A and $\widehat{A}$, which explains the role of Fourier-Mukai transform. In conclusion, it should be remarked that in our approach Theorem 1 is deduced, in a very indirect way, from the fact that it holds on abelian varieties. In fact Mumford's result (0.1) is the key ingredient of the Fourier-Mukai equivalence of categories. An interesting point of our methods is that –in essence – they use only the fact that the Fourier-Mukai transform is a fully faithful functor. This justifies the hope to generalize some of the present constructions and results to a non-abelian setting.

I warmly thank Christopher Hacon for kindly sending me his manuscript [H] on Green-Lazarsfeld's conjecture.

1. GLOBAL CO-GAUSSIAN MAPS

1.0. Introduction to §1-3. The purpose of this and the next two sections is to prove a quite general *first-order infinitesimal vanishing criterion* (Theorem 3.5 below). We start with a few words of motivation for the construction and the result, whose quite classical essence might be obscured by the details of the proofs. We illustrate the matter in the simplest case, i.e. the criterion for the vanishing of the H^1 of a line bundle Q on a smooth surface X. Let $C \subset X$ be a smooth curve such that $\mathcal{O}_X(C) \otimes Q$ is very ample and let $N = \mathcal{O}_C(C)$ be its normal bundle. We consider a very ample linear series $V \subset H^0(C, N \otimes Q)$ and the embedding $C \hookrightarrow \mathbf{P}(V)$. Letting $\mathcal{I}_C^V$ the ideal sheaf of C in $\mathbf{P}(V)$, from the exact sequence $0 \to \mathcal{I}_C^V / {\mathcal{I}_C^V}^2 \to \Omega^1_{\mathbf{P}(V)_{|C}} \to \Omega^1_C \to 0$ we get

the long exact sequence

$$\cdots \to \mathrm{Hom}(\Omega^1_{\mathbf{P}(V)|C}, N^\vee) \to \mathrm{Hom}(\mathcal{I}^V_C/{\mathcal{I}^V_C}^2, N^\vee) \stackrel{\psi_1^{P(V)}}{\to} \mathrm{Ext}^1(\Omega^1_C, N^\vee) \stackrel{\phi_1^{P(V)}}{\to} \mathrm{Ext}^1(\Omega^1_{\mathbf{P}(V)|C}, N^\vee) \to$$

If V is the complete linear series $H^0(C, N \otimes Q)$ we will suppress V in the notation, i.e. we will simply write $\mathbf{P}$, $\mathcal{I}_C$, ψ_1^P and ϕ_1^P. We recall that the dual of the map ϕ_1^P is known in the literature as a *gaussian map* ([BeEL],[W2], see also Remarks 1.5 and 3.9 below). We consider the natural exact sequence $0 \to N^\vee \to \Omega^1_{X|C} \to \Omega^1_C \to 0$. Its extension class represents a first-order deformation

$$e_C^X \in \mathrm{Ext}^1(\Omega^1_C, N^\vee).$$

In this case the co-gaussian vanishing criterion is stated as follows:

Theorem. *Assume that $C \sim kD$ with D ample and k big enough. Then: $e_C^X \in \ker \phi_1^P$ if and only if $H^1(X, Q) = 0$.*

Let us provide an informal motivation, building on methods and ideas of Beauville-Mérindol ([BM]), Voisin ([V]) and Reid ([R]). We consider the divisor $2C$ in X and the restriction map

$$\rho^{2C,C} : H^0(2C, \mathcal{O}_{2C}(C) \otimes Q) \to H^0(C, N \otimes Q).$$

The point is the following

Lemma. *Assume that $C \sim kD$ with D ample and k big enough. Then: $\rho^{2C,C}$ is surjective if and only if $e_C^X \in \ker(\phi)$.*

The Theorem is a corollary of the Lemma. Indeed an easy application of Serre's vanishing shows that, for k big enough, *infinitesimal completeness* i.e. the surjectivity of $\rho^{2C,C}$, is equivalent to honest *completeness*, i.e. the surjectivity of $\rho^{X,C} : H^0(X, \mathcal{O}_X(C) \otimes Q) \to H^0(C, N \otimes Q)$, which is in turn equivalent, again by Serre's vanishing, to the vanishing of $H^1(X, Q)$.

The Lemma is explained as follows. The double structure $2C$ induces, for $V = Im(\rho^{2C,C})$, an embedded first-order deformation $f_C^X \in \mathrm{Hom}(\mathcal{I}_C^V/{\mathcal{I}_C^V}^2, N^\vee)$ such that $\psi^V(f_C^X) = e_C^X$. This proves the direct implication of the Lemma since, if $Im(\rho^{2C,C})$ coincides with the complete linear series, then $e_X^C \in \ker(\phi_1^P)$ (this explains the relation with the extendibility problem). To prove the converse implication, which is the most important for our applications, let us assume that $e_C^X \in \ker(\phi_1^P)$. Then there is an first-order deformation $f \in \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, N^\vee)$ such that $\psi_1^P(f) = e_C^X$. The first-order deformation f is represented by a scheme $\widetilde{C}$ of multiplicity two on C embedded in $\mathbf{P}$ such that $\mathcal{I}_{C/\widetilde{C}} \cong N^\vee$. Therefore the infinitesimal completeness holds for $\widetilde{C}$, i.e. the restriction $\rho^{\widetilde{C},C} : H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(1)) \to H^0(C, N \otimes Q)$ is surjective. The first-order deformation f induces, by projection on $V := Im(\rho^{2C,C})$, a first-order deformation $f^V \in \mathrm{Hom}(\mathcal{I}_C^V/{\mathcal{I}_C^V}^2, N^\vee)$. Hence $f^V - f_C^X \in \ker(\psi_1^{P(V)})$. We assert that:

If $C \sim kD$, k is big enough and $V = Im(\rho^{2C,C})$, then $\psi_1^{P(V)}$ is "essentially injective", in the sense that $\ker \psi_1^{P(V)}$ consists only of linear changes of coordinates.

The converse part of the Lemma follows from the above assertion, since f_C^X will be projectively isomorphic to f^V and hence to f. This means that $2C$ is projectively isomorphic to f_C^X and therefore $2C$ itself satisfies the infinitesimal completeness condition, i.e. the surjectivity of $\rho^{2C,C}$, since f_C^X does.

The assertion above is seen as follows: $\ker\psi_1^{P(V)} = \mathrm{Hom}(\Omega^1_{\mathbf{P}(V)_{|C}}, N^\vee)$, which is computed by the cohomology of the Euler sequence:

$$\cdots \to V^\vee \otimes H^0(Q_{|C}) \to \mathrm{Hom}(\Omega^1_{\mathbf{P}(V)_{|C}}, N^\vee) \to \mathrm{Ext}^1(\mathcal{O}_C, N^\vee) \xrightarrow{c^V} V^\vee \otimes H^1(Q_{|C}) \to \cdots$$

We claim that if $k >>$ then c^V is injective. This proves the assertion since then $\mathrm{Hom}(\Omega^1_{\mathbf{P}(V)_{|C}}, N^\vee)$ is isomorphic to $V^\vee \otimes H^0(Q_{|C})$, which represents (unless $H^0(Q_{|C})$ is already zero) linear changes of coordinates. The injectivity of c^V is proved as follows: the dual of c^V is the multiplication map $m^V : V \otimes H^0(C, \omega_C \otimes Q^\vee) \to H^0(C, \omega_C \otimes N)$. Since V contains the restriction of $H^0(X, \mathcal{O}_X(C) \otimes Q)$ and $H^0(\omega_C \otimes Q^\vee)$ contains the restriction of $H^0(X, \omega_X(C) \otimes Q^\vee)$, it follows easily from Serre's vanishing that m^V is surjective for $C \sim kD$ and k big.

The content of the first three sections is, in essence, the following:

(a) we define *generalized co-gaussian maps* ϕ_n^P, where the curve C is replaced by a flag $C \subset Y$, $(n = \dim Y)$, where C is a multiple (1-dimensional) hyperplane section of Y. In this way we recover an analogous criterion for the vanishing of H^n of line bundle Q on a $n+1$-dimensional variety X (here the difficulty is that in codimension > 1 there is no immediate analogue of the divisor $2C$).

(b) We extend everything to a relative context via the notion of *global co-gaussian map* (denoted $\phi_n^{\mathcal{P}_T}$), in order to provide an analogous vanishing criterion for higher direct images of *projective families of line bundles* on a variety X. (Gaussian maps associated to *families* of line bundles were already studied, in a somewhat different context, in the previous work [P].)

1.1. Notation, set-up and assumptions. In this section we define *global co-gaussian maps* associated to a tuple $(Y, N, C, T, \mathcal{L})$ where:

- Y is a n-dimensional, irreducible Cohen-Macaulay variety and N is a very ample line bundle on Y;

- C is an irreducible curve in Y, complete intersection of $n-1$ divisors in the linear system $|N|$. If $n = 1$ we understand $C = Y$ with the choice of a very ample line bundle N on C;

- $(T, \mathcal{L})$ is a *projective family of line bundles* on Y, i.e. T is a projective scheme and $\mathcal{L}$ an invertible sheaf on $Y \times T$. We assume that $\mathcal{L}$ is *relatively base point-free*, i.e. that the evaluation map $ev_{\mathcal{L}} : p_T^*(p_{T_*}(\mathcal{L})) \to \mathcal{L}$ is surjective.

1.2. Definition. *(Global co-gaussian maps)* Let us denote respectively p_1, p_2 and $\Delta_C^{Y,T}$ the two projections of $(Y \times T) \times_T (C \times T)$ and the graph of the T-embedding $C \times T \hookrightarrow Y \times T$, i.e. the diagonal of $(C \times T) \times_T (C \times T)$ seen as a subscheme of $(Y \times T) \times_T (C \times T)$. We denote also p_Y the projection on the first factor of $Y \times T$. We have the restriction map

$$res : p_{1_*}(\mathcal{I}_{\Delta_C^{Y,T}} \otimes p_2^*(\mathcal{L}_{|C \times T})) \otimes \mathcal{L}^\vee \to p_{1_*}(\mathcal{I}_{\Delta_C^{Y,T}} \otimes \mathcal{O}_{\Delta_C^Y}) \cong p_Y^*(\Omega^1_{Y|C}) \tag{1.1}$$

The *global co-gaussian map* associated to the above data is obtained applying $\mathrm{Ext}^n_{Y \times T}(?, p_Y^* N^{-n})$ to (1.1) and restricting to $\mathrm{Ext}^n_Y(\Omega_{Y|C}, N^{-n}) \subset \mathrm{Ext}^n_{Y \times T}(p_Y^*(\Omega_{Y|C}), p_Y^* N^{-n})$:

$$\phi = \phi_n^{\mathcal{P}_T} : \mathrm{Ext}^n_Y(\Omega_{Y|C}, N^{-n}) \to \mathrm{Ext}^n_{Y \times T}(p_{1_*}(\mathcal{I}_{\Delta_C^{Y,T}} \otimes p_2^* \mathcal{L}_{|C \times T}), \mathcal{L} \otimes p_Y^* N^{-n}) \tag{1.2}$$

1.3. Remark. *(Elementary transformations)* The $\mathcal{O}_{Y \times T}$-module $p_{1_*}(\mathcal{I}_{\Delta_C^Y} \otimes p_2^*(\mathcal{L}_{|C \times T}))$ is the *elementary transformation* of $\mathcal{L}_{|C \times T}$, seen as a coherent sheaf on $Y \times T$. Following the notation of

Lazarsfeld, we will denote it $F_{Y\times T}^{\mathcal{L}_{|C\times T}}$. In fact $F_{Y\times T}^{\mathcal{L}_{|C\times T}}$ is the kernel of the evaluation map $ev_{\mathcal{L}_{|C\times T}}$:

$$0 \to F_{Y\times T}^{\mathcal{L}_{|C\times T}} \to p_T^*(p_{T_*}(\mathcal{L}_{|C\times T})) \stackrel{ev}{\to} \mathcal{L}_{|C\times T} \to 0 \qquad (1.3),$$

where p_T denotes the projection on the second factor of $Y \times T$. Sequence (1.3) follows from the standard exact sequence $0 \to \mathcal{I}_{\Delta_C^{Y,T}} \to \mathcal{O}_{(Y\times T)\times_T(C\times T)} \to \mathcal{O}_{\Delta_C^{Y,T}} \to 0$ using that, by flat base change, $p_{1_*}(p_2^*(\mathcal{L}_{|C\times T})) \cong p_T^*(p_{T_*}(\mathcal{L}_{|C\times T}))$.

1.4. Remark. *(Equivalent description of the domain)* We have the natural isomorphism

$$\mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C}) \stackrel{\sim}{\to} \mathrm{Ext}^n_Y(\Omega^1_{Y|C}, N^{-n}) \qquad (1.4)$$

defined by seeing $e \in \mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C})$ as an extension class on Y and composing it with the class in $\mathrm{Ext}^{n-1}_Y(N^{-1}_{|C}, N^{-n})$ of the Koszul resolution of $\mathcal{O}_C$ as $\mathcal{O}_Y$-module

$$0 \to N^{-n+1} \to .. \to \oplus N^{-1} \to \mathcal{O}_Y \to \mathcal{O}_C \to 0 \qquad (1.5)$$

twisted by N^{-1}. The fact that the map (1.4) is an isomorphism follows from the the change-of-ring spectral sequence $\mathrm{Ext}^p_C(?, \mathcal{E}xt^q_Y(\mathcal{O}_C, N^{-n})) \Rightarrow \mathrm{Ext}^{p+q}_Y(?, N^{-n})$. Indeed, since Y is Cohen-Macaulay, $\mathcal{E}xt^q_Y(\mathcal{O}_C, \mathcal{O}_Y) = N^{n-1}_{|C}$ if $q = n-1$ and zero otherwise, and therefore $\mathcal{E}xt^q_Y(\mathcal{O}_C, N^{-n}) = N^{-1}_{|C}$ if $q = n-1$ and zero otherwise.

In conclusion, the co-gaussian map (1.2) can be rewritten as follows

$$\phi_n^{\mathcal{P}_T} : \mathrm{Ext}^1_C(\Omega_{Y|C}, N^{-1}_{|C}) \to \mathrm{Ext}^n_{Y\times T}(F_{Y\times T}^{\mathcal{L}_{|C\times T}}, \mathcal{L} \otimes p_Y^* N^{-n}) \qquad (1.6)$$

1.5. Remark. *(Gaussian maps)* The terminology adopted in the definition above is due to the fact that the above maps belong to the family of *gaussian maps*, studied systematically by J. Wahl and other authors in the context of the *extendibility problem* (see e.g. [W1], [W2], [Z], [BeEL], [BM], [V] and Remark 3.9 below) They can be defined as follows: let C be a curve and let Δ, p_1 and p_2 be respectively the diagonal and the projections of $C \times C$. Given two line bundles L and M on C the associated gaussian map is:

$$\gamma_{L,M} := H^0(res_\Delta) : H^0(C, p_{1_*}(\mathcal{I}_\Delta \otimes p_2^* L) \otimes M) \to H^0(C, \Omega^1_C \otimes L \otimes M) \qquad (1.7)$$

Going back to Definition 1.2, take $Y = C$ and, as a family of line bundles, a single line bundle L parametrised by a simple point. By Serre duality the co-gaussian map $\phi_1^P : \mathrm{Ext}^1(\Omega^1_C, N^{-1}) \to \mathrm{Ext}^1(p_{1_*}(\mathcal{I}_\Delta \otimes p_2^* L), L \otimes N^{-1})$ is the dual of the gaussian map $\gamma_{N\otimes Q, \omega_C \otimes Q^{-1}}$, where $Q = L \otimes N^{-1}$.

2. THE BASIC PROPERTY OF CO-GAUSSIAN MAPS

The relevance of global co-gaussian maps in the context of vanishing theorems relies on Lemma 2.3 below, which, joined with Prop. 3.2, supplies a necessary condition for vanishing.

2.1. Assumptions. We keep all the notation, assumptions and set-up of the previous section. Moreover:

(a) we assume that Y is a (very ample) Cartier divisor of an irreducible locally Cohen-Macaulay variety X in such a way that $N = \mathcal{O}_X(Y)_{|Y}$; we assume also that the line bundle $\mathcal{L}$ is the restriction of a line bundle $\widetilde{\mathcal{L}}$ on $X \times T$;

(b) we furthermore assume that

$$R^i p_{T_*}(\mathcal{L} \otimes p_Y^*(N^{-n})) = 0 \qquad \text{for any } i < n \tag{2.1}$$

As the accurate reader will notice, assumption (2.1) is not really necessary for the results below. However, as we will see, in the applications we have in mind it is automatically satisfied, and it is helpful in simplifying the statements and some diagrams.

2.2. Notation. *(a) The connecting map.* We consider the divisor $2Y$ on X. Since $\mathcal{I}_{Y/2Y} \cong N^{-1}$ we have the short sequence

$$0 \to N^{-1} \to \mathcal{O}_{2Y} \to \mathcal{O}_Y \to 0.$$

Tensoring with $\mathcal{O}_X((-n+1)Y)$ we get

$$0 \to N^{-n} \to \mathcal{O}_{2Y}((-n+1)Y) \to N^{-n+1} \to 0$$

which, composed with the Koszul resolution of $\mathcal{O}_C$ as $\mathcal{O}_Y$-module (1.5), gives rise to the exact complex

$$0 \to N^{-n} \to \mathcal{O}_{2Y}((-n+1)Y)) \to \oplus N^{-n+2} \to \cdots \to \oplus N^{-1} \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$$

Applying the functor $p_{T_*}(\widetilde{\mathcal{L}} \otimes p_X^* ?)$ one gets the connecting map

$$\delta : p_{T_*}(\mathcal{L}_{|C\times T}) \to R^n p_{T_*}(\mathcal{L} \otimes p_Y^*(N^{-n})) \tag{2.2}$$

(b) The extension class. We denote

$$e = e_C^{X,Y} \in \mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C}) \overset{(1.4)}{\cong} \mathrm{Ext}^n_Y(\Omega^1_{Y|C}, N^{-n}). \tag{2.3}$$

the extension class of the exact sequence $0 \to N^{-1}_{|C} \to \Omega^1_{X|C} \to \Omega^1_{Y|C} \to 0$.

(c) The basic diagram. We have the diagram with exact column

$$\begin{array}{ccc} & & \mathrm{Ext}^n_{Y\times T}(\mathcal{L}_{|C\times T}, \mathcal{L} \otimes p_Y^*(N^{-n})) \\ & & \downarrow \alpha \\ & & \mathrm{Hom}(p_{T_*}(\mathcal{L}_{|C\times T}), R^n p_{T_*}(\mathcal{L} \otimes p_Y^*(N^{-n})) \\ & & \downarrow \beta \\ \mathrm{Ext}^1_C(\Omega_{Y|C}, N^{-1}_{|C}) & \xrightarrow{\phi} & \mathrm{Ext}^n_{Y\times T}(F_{Y\times T}^{\mathcal{L}_{|C\times T}}, \mathcal{L} \otimes p_Y^*(N^{-n})) \end{array} \tag{2.4}$$

where $\phi = \phi_n^{\mathcal{P}_T}$ is the co-gaussian map defined in (1.6) and the column is obtained applying $\mathrm{Ext}^n_{Y\times T}(?, \mathcal{L} \otimes p_Y^*(N^{-n}))$ to sequence (1.3). Here we are using the canonical isomorphism

$$\mathrm{Ext}^n_{Z\times T}(p_T^*(p_{T_*}(\mathcal{L}_{|C\times T})), \mathcal{L} \otimes p_Y^*(N^{-n})) \cong \mathrm{Hom}(p_{T_*}(\mathcal{L}_{|C\times T}), R^n p_{T_*}(\mathcal{L} \otimes p_Y^*(N^{-n})). \tag{2.5}$$

which follows from assumption (2.1), via degeneration of the adjunction spectral sequence $\mathrm{Ext}^p_T(p_{T_*}(\mathcal{L}_{|C\times T}), R^q p_{T_*}(?)) \Rightarrow \mathrm{Ext}^{p+q}_{Z\times T}(p_T^*(p_{T_*}(\mathcal{L}_{|C\times T})), ?)$.

Lemma 2.3 below describes a basic relation between the co-gaussian image $\phi(e)$ and the map δ.

2.3 Lemma. *$\beta(\delta) = \phi(e)$. Therefore, if $\delta = 0$ then $e \in \ker \phi$.*

Proof. Following Voisin's ideas ([V]), we describe $\phi(e)$ in terms of the elementary transformation of $\mathcal{L}_{|C\times T}$ seen as a sheaf on the scheme $2Y \times T$. To construct such elementary transformation we proceed as in the previous section. We denote p_1, p_2 and Δ_C^{2Y} respectively the two projections of $(2Y \times T) \times_T (C \times T)$ and the graph of the T-embedding $C \times T \hookrightarrow 2Y \times T$. We denote also p_{2Y} (resp. p_Y) the projections on the first factor of $2Y \times T$ (resp. $Y \times T$). We have the restriction map

$$res : p_{1_*}(\mathcal{I}_{\Delta_C^{2Y}} \otimes p_2^*(\mathcal{L}_{|C\times T})) \otimes \widetilde{\mathcal{L}}^\vee \to p_{1_*}(\mathcal{I}_{\Delta_C^{2Y}} \otimes \mathcal{O}_{\Delta_C^{2Y}}) \cong p_{2Y}^*(\Omega^1_{2Y|C}) \cong p_{2Y}^*(\Omega^1_{X|C}).$$

We denote:

$$F_{2Y\times T}^{\mathcal{L}_{|C\times T}} = p_{2_*}(\mathcal{I}_{\Delta_C^{2Y}} \otimes p_1^*(\mathcal{L}_{|C\times T})), \tag{2.6}$$

which can be seen, as in Remark 1.3, as an elementary transformation:

$$0 \to F_{2Y\times T}^{\mathcal{L}_{|C\times T}} \to p_T^*(p_{T_*}(\mathcal{L}_{|C\times T})) \xrightarrow{ev} \mathcal{L}_{|C\times T} \to 0 \tag{2.7}$$

Note that, by abuse of language, here and in (2.12) below we are denoting p_T both the projections on the second factor of $2Y \times T$ and $Y \times T$. Restricting (2.7) to $Y \times T$ one gets the exact sequence

$$0 \to p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L} \to F_{2Y\times T}^{\mathcal{L}_{|C\times T}}{}_{|Y\times T} \to F_{Y\times T}^{\mathcal{L}_{|C\times T}} \to 0 \tag{2.8}$$

(since $\mathcal{T}or_1^{2Y\times T}(\mathcal{O}^{2Y}_{\Delta_C}, \mathcal{O}_{Y\times T}) \cong p_Y^* N_{|C}^{-1})$). We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L} & \to & F_{2Y\times T}^{\mathcal{L}_{|C\times T}}{}_{|Y\times T} & \to & F_{Y\times T}^{\mathcal{L}_{|C\times T}} & \to 0 \\ & \| & & \downarrow res & & \downarrow res & \\ 0 \to & p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L} & \to & p_Y^*(\Omega^1_{X|C}) \otimes \mathcal{L} & \to & p_Y^*(\Omega^1_{Y|C}) \otimes \mathcal{L} & \to 0 \end{array}. \tag{2.9}$$

On the other hand one has the canonical inclusion (as direct summand)

$$k \cong \mathrm{Hom}(p_Y^* N_{|C}^{-1}, p_Y^* N_{|C}^{-1}) \overset{\Phi}{\hookrightarrow} \mathrm{Ext}^{n-1}_{Y\times T}(p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L}, p_Y^*(N^{-n}) \otimes \mathcal{L}) \tag{2.10}$$

obtained as follows: as for (1.4) one has the canonical isomorphism

$$k \cong \mathrm{Hom}(p_Y^* N_{|C}^{-1}, p_Y^* N_{|C}^{-1}) \cong \mathrm{Ext}^{n-1}_Y(p_Y^* N_{|C}^{-1}, p_Y^* N^{-n}) \tag{2.11}$$

obtained composing with (1.5) (the Koszul resolution of $\mathcal{O}_C$ as a $\mathcal{O}_Y$-module). The inclusion (2.10) follows since the target of (2.11) is naturally a direct summand of $\mathrm{Ext}^{n-1}_{Y\times T}(p_Y^* N_{|C}^{-1}, p_Y^* N^{-n})$. In conclusion we have proved:

2.4. Claim. *$\phi(e)$ is the image of $id \in \mathrm{Hom}(p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L}, p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L})$ via the composition of the inclusion Φ of (2.10) and the connecting map (from the top row of (2.9))*

$$\mathrm{Ext}^{n-1}_{Y\times T}(p_Y^*(N_{|C}^{-1}) \otimes \mathcal{L}, p_Y^*(N^{-n}) \otimes \mathcal{L}) \to \mathrm{Ext}^n(F_{Y\times T}^{\mathcal{L}_{|C\times T}}, p_Y^*(N^{-n}) \otimes \mathcal{L}).$$

We have seen the role of $F^{\mathcal{L}_{|C\times T}}_{2Y\times T}{}_{|Y\times T}$. Next, we turn to $F^{\mathcal{L}_{|C\times T}}_{2Y\times T}$ itself. By Snake's Lemma we have the commutative exact diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 & & & \mathcal{L}_{|C\times T} & = & \mathcal{L}_{|C\times T} & \\
 & & & \uparrow & & \uparrow & \\
0\to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T}))\otimes p_Y^*N^{-1} & \to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T})) & \to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T}))_{|Y\times T} & \to 0\\
 & \| & & \uparrow & & \uparrow & \\
0\to & \ker & \to & F^{\mathcal{L}_{|C\times T}}_{2Y\times T} & \to & F^{\mathcal{L}_{|C\times T}}_{Y\times T} & \to 0\\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array} \tag{2.12}$$

Diagram (2.9) and (2.12) fit together in the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0\to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T}))\otimes p_Y^*N^{-1} & \to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T})) & \to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T}))_{|Y\times T} & \to 0\\
 & \| & & \uparrow & & \uparrow & \\
0\to & p_T^*(p_{T_*}(\mathcal{L}_{|C\times T}))\otimes p_Y^*N^{-1} & \to & F^{\mathcal{L}_{|C\times T}}_{2Y\times T} & \to & F^{\mathcal{L}_{|C\times T}}_{Y\times T} & \to 0\\
 & \downarrow & & \downarrow & & \| & \\
0\to & \mathcal{L}\otimes p_Y^*(N^{-1}_{|C}) & \to & F^{\mathcal{L}_{|C\times T}}_{2Y\times T}{}_{|Y\times T} & \to & F^{\mathcal{L}_{|C\times T}}_{Y\times T} & \to 0
\end{array} \tag{2.13}$$

where the bottom half is (2.9) and the top half is (2.12). Claim 2.4 states that $\phi(e)$ is – via the inclusion Φ – the image of the identity map via the connecting map –for the appropriated Ext groups – of the base of the most external rectangle of (2.13). On the other hand, by commutativity of diagram (2.13), one can follow the other three edges of the external rectangle. The connecting map on the right vertical edge is the map β. An easy verification shows that the map δ is the image of the identity map, via the composition of the connecting maps on the left and the top horizontal edges (the details are left to the reader). This proves the Lemma.

3. CO-GAUSSIAN VANISHING CRITERION

This section is devoted to the infinitesimal vanishing criterion (Theorem 3 of the introduction, Theorem 3.5 below). Its content is that the necessary condition for vanishing supplied by the combination of Lemma 2.3 and Proposition 3.2 is also sufficient.

3.1. Notation and set-up. We continue with the assumptions, set-up and notation of the previous section, with two specifications:

(a) the Cartier divisor Y of the $(n+1)$-dimensional variety X is now assumed to be linearly equivalent to kD, with D ample and k a positive integer.

(b) We have a projective family of line bundles on X

$$(T, \mathcal{Q}) \tag{3.1}$$

and we apply the machinery of the previous section to

$$\mathcal{L} = \mathcal{Q}\otimes p_Y^* N^n, \tag{3.2}$$

(recall that N is the normal bundle of Y in X), which is the restriction to Y of the family $\widetilde{\mathcal{L}} = \mathcal{Q} \otimes p_X^* \mathcal{O}_X(nY)$ of line bundles on X. In Notation 2.2(a) we introduced the connecting map $\delta : p_{T_*}(p_Y^* N^n_{|C} \otimes \mathcal{Q}) \to R^n p_{T_*}(\mathcal{Q}_{|Y\times T})$. We begin by pointing out a basic property of such map

3.2 Proposition. *Assume that: (i) $k >>$; (ii) $R^i p_{T_*}(\mathcal{Q}) = 0$ for any $i < n$. Then: $R^n p_{T_*}(\mathcal{Q}) = 0$ if and only if $\delta = 0$.*

Proof. Let us consider the case $n = 1$ first. Here $\delta = 0$ simply means that the restriction map $p_{T_*}(\mathcal{Q} \otimes p_X^*(\mathcal{O}_{2Y}(Y)) \to p_{T_*}(\mathcal{Q} \otimes p_Y^* N)$ is surjective (infinitesimal completness). Since, by relative Serre vanishing and Serre duality, the restriction map $p_{T_*}(\mathcal{Q} \otimes p_X^*(\mathcal{O}_X(Y)) \to p_{T_*}(\mathcal{Q} \otimes p_X^*(\mathcal{O}_{2Y}(Y))$ is surjective, it turns out that infinitesimal completeness is equivalent to honest completness, i.e. the surjectivity of the restriction map $p_{T_*}(\mathcal{Q} \otimes p_X^*(\mathcal{O}_X(Y)) \to p_{T_*}(\mathcal{Q} \otimes p_Y^* N)$. This, again by Serre vanishing, is in turn equivalent to the vanishing of $R^n p_{T_*}(\mathcal{Q})$. This concludes the case $n = 1$.
$n > 1$: The connecting map δ of the statement factors as $\delta'' \circ \delta'$, where

$$\delta' : p_{T_*}(\mathcal{Q} \otimes p_Y^* N^n_{|C})) \to R^{n-1} p_{T_*}(\mathcal{Q} \otimes p_Y^* N) \quad \text{and} \quad \delta'' : R^{n-1} p_{T_*}(\mathcal{Q} \otimes p_Y^* N) \to R^n p_{T_*}(\mathcal{Q}_{|Y\times T}). \tag{3.3}$$

δ' is the connecting map of the exact complex (of $\mathcal{O}_{Y\times T}$-modules)

$$0 \to \mathcal{Q} \otimes p_Y^* N \to \oplus \mathcal{Q} \otimes p_Y^* N^2 \to \cdots \to \oplus \mathcal{Q} \otimes p_Y^* N^{n-1} \to \mathcal{Q} \otimes p_Y^* N^n \to \mathcal{Q} \otimes p_Y^* N^n_{|C} \to 0. \tag{3.4}$$

and δ'' is the connecting map of the exact sequence

$$0 \to \mathcal{Q}_{|Y\times T} \to \mathcal{Q} \otimes p_X^*(\mathcal{O}_{2Y}(Y)) \to \mathcal{Q} \otimes p_Y^* N \to 0 \tag{3.5}$$

3.3 Claim. *For $k >>$, the map δ' is zero if and only if its target is zero.*

Proof. From the exact sequence

$$0 \to \mathcal{Q} \otimes p_X^* \mathcal{O}_X((i-1)Y) \to \mathcal{Q} \otimes \mathcal{O}_X(iY) \to \mathcal{Q} \otimes N^i \to 0$$

it follows that $R^h p_{T_*}(\mathcal{Q} \otimes N^i) = 0$ for any h, i such that $h > 0$ and $i > 1$, since, by relative Serre's vanishing, $R^h p_{T_*}(\mathcal{Q} \otimes p_X^* \mathcal{O}_X(iY)) = R^{h+1} p_{T_*}(\mathcal{Q} \otimes p_X^* \mathcal{O}_X((i-1)Y)) = 0$. Then Claim 3.3 follows immediately by chopping (3.4) into short exact sequences.

3.4 Claim. *If $k >>$ then the map δ'' is injective.*

Proof. By relative Serre duality and relative Serre vanishing we have that

$$R^{n-1} p_{T_*}(\mathcal{Q} \otimes p_X^* \mathcal{O}_X(Y)) = R^n p_{T_*}(\mathcal{Q} \otimes p_X^* \mathcal{O}_X(-Y)) = 0$$

and this, by the sequence

$$0 \to \mathcal{Q} \otimes p_X^* \mathcal{O}_X(-Y) \to \mathcal{Q} \otimes p_X^* \mathcal{O}_X(Y) \to \mathcal{Q} \otimes p_X^* \mathcal{O}(Y)_{|2Y\times T} \to 0,$$

yields that $R^{n-1} p_{T_*}(\mathcal{Q} \otimes \mathcal{O}_X(Y)_{|2Y\times T}) = 0$ for $k >>$.

From the Claim 3.3 and 3.4 it follows that $\delta = 0$ if and only if the target of δ', i. e. $R^{n-1} p_{T_*}(\mathcal{Q} \otimes p_Y^* N)$, vanishes and this, again by relative Serre vanishing, is equivalent to of $R^n p_{T_*}(\mathcal{Q}) = 0$. This proves the Proposition.

We come to the main result. We consider the co-gaussian map $\phi = \phi_n^{\mathcal{P}_T}$ associated to the data $(Y, N, C, T, p_Y^* N^n \otimes \mathcal{Q})$ and let $e = e_C^{X,Y} \in \mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C})$ be the extension class of Notation 2.2(b).

3.5. Theorem *(First-order infinitesimal vanishing criterion) Assume that: (i) $k >>$; (ii) $R^i p_{T_*}(\mathcal{Q}) = 0$ for any $i < n$. Then: $R^n p_{T_*}(\mathcal{Q}) = 0$ if and only if $e \in \ker(\phi)$.*

Proof. Note that we can use Lemma 2.3 since the assumptions made in 1.1 and 2.1 are fulfilled by the family of line bundles $(T, \mathcal{L}) = (T, \mathcal{Q} \otimes p_Y^* N^n)$ for $k >>$. In fact, in the first place, if $k >>$ then $\mathcal{Q} \otimes p_X^* N^n$ is relatively base point-free, by relative Serre Theorem. Furthermore Assumption 2.1, i.e. $R^i p_{T_*}(\mathcal{Q}_{|Y\times T}) = 0$ for $i \neq n$ is achieved, for $k >>$, because of hypothesis (ii) of the present statement and Serre vanishing. Therefore the direct implication follows at once from Proposition 3.2 and Lemma 2.3. To prove the converse, let us consider the co-multiplication map

$$\alpha : \mathrm{Ext}^n_{Y\times T}(p_Y^* N^n_{|C}, \mathcal{O}_{Y\times T}) \to \mathrm{Hom}(p_{T_*}(\mathcal{Q} \otimes p_Y^* N^n_{|C}), R^n p_{T_*}(\mathcal{Q}_{|Y\times T})) \tag{3.6}$$

and the induced co-multiplication map

$$\alpha' : \mathrm{Ext}^n_{Y\times T}(p_Y^* N^n_{|C}, \mathcal{O}_{Y\times T}) \to \mathrm{Hom}(p_{T_*}(\mathcal{Q} \otimes p_Y^* N^n), R^n p_{T_*}(\mathcal{Q}_{|Y\times T})) \tag{3.7}$$

The map α, which appears in the basic diagram (2.4) of Notation 2.2(c), takes the extension class of the exact complex

$$0 \to \mathcal{O}_{Y\times T} \to E_n \to \cdots \to E_1 \to p_Y^* N^n_{|C} \to 0,$$

tensored with $\mathcal{Q}$, to the corresponding connecting map $d : p_{T_*}(\mathcal{Q} \otimes p_Y^*(N^n_{|C})) \to R^n p_{T_*}(\mathcal{Q}_{|Y\times T})$ while α' takes the same extension class to the map d composed with the restriction map $res : p_{T_*}(\mathcal{Q} \otimes p_Y^* N^n) \to p_{T_*}(\mathcal{Q} \otimes p_Y^* N^n_{|C})$. We consider also the maps α_Y (resp. α'_Y) obtained restricting the maps α (resp. α') to the direct summand $\mathrm{Ext}^n_Y(N^n_{|C}, \mathcal{O}_Y)$. The Theorem follows at once from the steps below, combined with Prop. 3.2:

3.6 Claim. *Assume that the map α'_Y is injective and that $e \in \ker(\phi)$. Then $\delta = 0$.*

Proof. By Lemma 2.3 if $e \in \ker\phi$ then δ lies in the kernel of the map β of (2.4) i.e. it is the image of an element u via the co-multiplication map α. In fact it follows easily that one can assume that u lies in the direct summand $\mathrm{Ext}^n_Y(N^n_{|C}, \mathcal{O}_Y)$. On the other hand, by its very definition, δ composed with the restriction map res is zero. It follows that $u \in \ker\alpha'_Y$. Therefore the assumption yields $u = 0$. Hence δ is zero.

3.7 Claim. *If $k >>$ the map α'_Y is injective.*

Proof. There is the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_Y(N^n_{|C}, \mathcal{O}_Y) & \stackrel{\alpha'_Y}{\to} & \mathrm{Hom}(p_{T*}(\mathcal{Q} \otimes p_Y^* N^n), R^n p_{T_*}(\mathcal{Q}_{|Y\times T})) \\
\| & & \\
\mathrm{Ext}^{n+1}_X(N^n_{|C}, \mathcal{O}_X(-Y)) & & \downarrow c \\
\downarrow a & & \\
\mathrm{Ext}^{n+1}_X(\mathcal{O}_X(nY), \mathcal{O}_X(-Y)) & \stackrel{b_X}{\to} & \mathrm{Hom}(p_{T*}(\mathcal{Q} \otimes p_X^* \mathcal{O}_X(nY)), R^{n+1} p_{T_*}(\mathcal{Q} \otimes p_X^* \mathcal{O}_X(-Y)))
\end{array}$$

where: a and c are the natural maps, b_X is the co-multiplication map defined as usual and the vertical isomorphism is proved as for (1.4). Therefore, to prove the injectivity of α' it is enough

to show the injectivity of a and b_X. We prove only the injectivity of b_X (the same question for a, which is much easier, is left to the reader). For $t \in T$ let us denote Q_t the line bundle on X parametrised by t, i.e. $\mathcal{Q}_{|X\times t}$ and the co-multiplication map

$$b_X^t : \mathrm{Ext}^{n+1}_X(\mathcal{O}_X(nY), \mathcal{O}_X(-Y)) \to \mathrm{Hom}_k(H^0(Q_t \otimes p_X^*\mathcal{O}_X(nY)), H^{n+1}(Q_t \otimes p_X^*\mathcal{O}_X(-Y))) \quad (3.8)$$

As we are assuming $k >>$ it follows that the target of b_X^t is naturally identified to the fibre at t of the locally free sheaf $\mathcal{F} := \mathcal{H}om(p_{T*}(\mathcal{Q} \otimes p_X^*\mathcal{O}_X(nY)), R^{n+1}p_{T_*}(\mathcal{Q} \otimes p_X^*\mathcal{O}_X(-Y))$ and that the the map b_X^t is the composition of the map b_X and the evaluation at t of the global sections of $\mathcal{F}$. Therefore, to prove that there exists a k_0 such that b_X is injective for any $k \geq k_0$ it is enough to show that, for some $t \in T$, there is a $k_0(t)$ such that the map b_X^t is injective for any $k \geq k_0(t)$. But this follows since b_X^t is the dual of the multiplication map of global sections

$$m_t : H^0(Q_t \otimes p_X^*\mathcal{O}_X(nY)) \otimes H^0(Q_t^\vee \otimes \omega_X \otimes \mathcal{O}_X(Y)) \to H^0(\omega_X \otimes \mathcal{O}_X((n+1)Y))$$

which is easily seen to be surjective for $k >>$ (by Serre vanishing on $X \times X$). The details are left to the reader.

3.8. Remark. Theorem 3.5 in fact yields a vanishing criterion also for the other higher direct images $R^ip_{T_*}(\mathcal{Q})$, $0 < i < n$. Indeed, let H be an ample divisor on X and let Z be a $i+1$-dimensional, reduced and irreducible, complete intersection of divisors in the linear system $|mH|$. It is easy to see, arguing as usual with relative Serre vanishing, that $R^ip_{T_*}(\mathcal{Q}) \cong R^ip_{T_*}(\mathcal{Q}_{|Z\times T})$ if $m >>$. Then one can apply Theorem 3.2 to Z and the family of line bundles $(T, \mathcal{Q}_{|Z\times T})$.

3.9. Remark. *(Relation with the approach to the extendibility problem via gaussian maps)* Continuing with the theme of Remark 1.5 and of the Motivation 0.1, we relate the present co-gaussian criterion with the main theorem about gaussian maps and the extendibility problem, namely that (under some assumptions) *if a curve C – embedded by a complete linear series $|N|$ in projective space – is a hyperplane section then the gaussian map γ_{N,ω_C} is non-surjective* ([W],[BM], see also [Z] for a more general result, as explained in [BeEL]). In fact, according to Beauville-Mérindol treatment [BM], the essence of such result is that if $C \subset \mathbf{P}(H^0(N)) = \mathbf{P}^r$ is a hyperplane section of $X \subset \mathbf{P}^{r+1}$ then the class of the cotangent extension $e_C^X \in \mathrm{Ext}^1(\Omega^1_C, N^\vee)$ is contained in the kernel of the dual of γ_{N,ω_C}. This yields that if C is a divisor of a *regular* surface X, i.e. such that $H^1(\mathcal{O}_X) = 0$, then e_X^C lies in the kernel of the dual of γ_{N,ω_C}. This gives a non-trivial condition on provided that $e_C^X \neq 0$, which is almost always the case (*loc cit*). The direct implication of Theorem 3.1 can be seen as a generalization of the previous statement in three different directions: (1) *line bundles different from $\mathcal{O}_X$* . In fact, using Remark 1.5, Theorem 3.5 says that (under some assumptions), given a line bundle Q on a surface X, if $H^1(X,Q) = 0$ then e_C^X lies in the kernel of the dual of the gaussian map $\gamma_{N\otimes Q,\omega_C\otimes Q^\vee}$. (2) *Higher dimensional varieties and higher cohomology groups*; (3) *Families* of line bundles and *higher direct images.* We stress that the condition $k >>$ of Theorem 3.5 is not relevant for the direct implication, and it can be removed, even if it helps in simplifying some proofs in the relative setting. We hope that such generalizations will have an independent interest.

On the other hand, the most important part for the applications of the present paper is that there is a partial converse to extendibility, namely the converse implication of Theorem 3.5, which definitely needs the assumption $k >>$ (which, however, can be made effective).

4. MAIN STATEMENTS

The purpose of this section is to state precisely Theorem 4 of the introduction (Theorem 4.5 below) and show how it implies our version of Green-Lazarsfeld's conjecture on vanishing of higher direct images of Poincarè line bundles (Theorem 4.7 below).

4.1. Set-up. Let $a_Y : Y \to A$ be a morphism from an irreducible, n-dimensional Gorenstein variety to an abelian variety. Let $\mathcal{Q}_{Y\times\widehat{A}}$ be the pullback of a Poincarè line bundle $\mathcal{P}$ on $A\times\widehat{A}$: $\mathcal{Q}_{Y\times\widehat{A}} = (a_Y \times id_{\hat{A}})^*(\mathcal{P})$. Then we take as data to define a global co-gaussian map (see Notation, Set-up and Assumptions 1.1.) the tuple $(Y, N, C, \widehat{A}, p_Y^* N^n \otimes \mathcal{Q}_{Y\times\widehat{A}})$. Therefore it is defined the co-gaussian map

$$\phi_n^{\mathcal{P}_{\hat{A}}} : \mathrm{Ext}^1_C(\Omega^1_{Y\,|C}, N^{-1}_{|C}) \to \mathrm{Ext}^n_{Y\times\hat{A}}(p_{1_*}(\mathcal{I}_{\Delta_C^{Y,\hat{A}}} \otimes p_2^*(p_Y^*(N^n_{|C}) \otimes \mathcal{Q}_{Y\times\hat{A}})), \mathcal{Q}_{Y\times\hat{A}}) \tag{4.1}$$

Our goal is, roughly speaking, to compare certain first-order deformations of the map $a_{|C} : C \to A$ with certain relative first-order embedded deformations of the flag $C\times\widehat{A} \subset Y\times\widehat{A}$ inside the "projective bundle" $\mathcal{P}_{\widehat{A}} = \mathbf{P}(p_{\hat{A}*}(p_C^*(N^n_{|C})\otimes\mathcal{Q}))$. Since the global co-gaussian map takes care of the latter, it is natural to consider a sort of analog of co-gaussian maps associated to the morphism $a_{|C} : C \to A$. This is defined in 4.2 below. In 4.3 it is shown the analogy with co-gaussian maps.

4.2 Definition. (Co-multiplication maps) We consider the restricted differential of the morphism a_Y: $(d_{a_Y})_{|C} : T_{Y\,|C} \to T_{A.0}\otimes\mathcal{O}_C$. Applying $H^1(C, N^{-1}_{|C}\otimes ?)$ we get the map

$$\phi^A : H^1(T_{Y\,|C}\otimes N^{-1}_{|C}) \to T_{A,0}\otimes H^1(N^{-1}_{|C}). \tag{4.2}$$

4.3. The map ϕ^A_n goes as follows: let $\tilde{\phi}^A_n : \mathrm{Ext}^1_C(\Omega^1_{Y\,|C}, N^{-1}_{|C}) \to \mathrm{Hom}_k(\Omega_{A.0}, H^1(C, N^{-1}_{|C})$ be the map obtained by applying $\mathrm{Ext}^1_C(?, N^{-1}_{|C})$ to the restricted co-differential $\Omega^1_{A,0_{|C}} \to \Omega^1_{Y\,|C}$. (Note that $\tilde{\phi}^A_1$ is perfectly analogous to the "simple" co-gausssian map Φ^P defined in 1.0. Here we have to pass to the map ϕ^A_1 because of the fact that $a_{|C}$ is not necessarily an embedding.) We have that $\tilde{\phi}^A_n$ is a co-multiplication map, in the sense that it takes the extension class e of an exact sequence $0 \to N^{-1}_{|C} \to E \to \Omega^1_{Y\,|C} \to 0$ to the morphism $\rho_e : \Omega^1_{A,0} \to H^1(N^{-1}_{|C})$, composition of the coboundary map $H^0(\Omega^1_{Y\,|C}) \to H^1(N^{-1}_{|C})$ with the map $\Omega^1_{A,0} \to H^0(\Omega^1_{Y\,|C})$. Thus ϕ^A_n is the restriction of $\tilde{\phi}^A$ to the subspace $\mathrm{Ext}^1(N, T_{Y\,|C}) \subset \mathrm{Ext}^1_C(\Omega^1_{Y\,|C}, N^{-1}_{|C})$, where the inclusion takes the extension class of $0 \to T_{Y\,|C} \to F \to N \to 0$ to the class of the dual

$$0 \to N^{-1}_{|C} \to E \to \Omega^{1\vee\vee}_{Y}{}_{|C} \to 0 \tag{4.3}$$

which is seen as an element of $\mathrm{Ext}^1_C(\Omega^1_{Y\,|C}, N^\vee)$ via the natural map $\Omega^1_{Y\,|C} \to \Omega^{1\vee\vee}_{Y}{}_{|C}$.
We now introduce a third linear map, which takes care of the rank of the morphism $a_Y : Y \to A$.

4.4. Definition. (Induced map) In the set-up of Definition 4.2., we consider the map

$$\Phi^A_n : \ker(\phi^A_n) \to \mathrm{Hom}_k(\Lambda^{n+1}\Omega^1_{A,0}\,, H^0(N^{-1}\otimes\omega_{Y\,|C})) \tag{4.4}$$

defined as follows: given a sequence as (4.3) such that its extension class e sits in $\ker(\phi^A_n)$, we have the induced evaluation map $ev_E : \Omega^1_{0,A}\otimes\mathcal{O}_C \to E$. The map

$$H^0(\Lambda^{n+1}ev_E) : \Lambda^{n+1}\Omega^1_{A,0} \to H^0(N^{-1}\otimes\omega_{Y\,|C}), \tag{4.5}$$

(which depends only on e) is, by definition, $\Phi^A_n(e)$.

4.5. Theorem. *Let $(Y,N,C,\widehat{A},\mathcal{Q}_{Y\times\widehat{A}})$ be as in 4.1. If: (i) the map $a_{|C}:C\to A$ is birational onto its image, and (ii) the map Φ^A_n is non zero, then:* $\ker(\phi^A_n)\subset\ker(\phi^{\mathcal{P}_{\hat{A}}}_n)$.

Note that hypothesis (ii) means that there is a class $e\in\ker(\phi^A_n)$ such that the evaluation map $ev_E:\Omega^1_{A,0}\otimes\mathcal{O}_C\to E$ is generically surjective.

The main Theorem of the paper (Theorem 1 of the introduction, Theorem 4.7 below) is a corollary of Theorems 3.2 and 4.5.

Theorem 4.6. *Let $a:X\to A$ be a separable and generically finite morphism from an irreducible, Gorenstein, projective variety to an abelian variety. Then $R^ip_{A^\vee_*}((a,id_{\hat{A}})^*\mathcal{P})=0$ for any $i<\dim X$.*

Proof of Theorem 4.6. Induction on $\dim X$. For $\dim X=1$ the statement is trivial. Assume $\dim X>1$. Let D be an ample Cartier divisor on X and let Y be a reduced, irreducible Gorenstein divisor in the linear system $|kD|$. If $k>>$ the restricted morphism $a_{|Y}$ is separable and generically finite too. Hence, by inductive hypothesis, $R^ip_{A^\vee_*}((a_{|Y},id_{\hat{A}})^*\mathcal{P})=0$ for $i\le\dim Y-1$. Since, as $m>>$, $R^ip_{A^\vee_*}((a,id_{\hat{A}})^*\mathcal{P})\cong R^ip_{A^\vee_*}((a_{|Y},id_{\hat{A}})^*\mathcal{P})$ for any $i<\dim X-1$ (relative Serre vanishing), it remains to prove the statement only for $i=\dim X-1$. We choose, a usual, a curve $C\subset Y$ complete intersection of $\dim X-1$ divisors linearly equivalent to Y. As the positive integer k is big enough the flag $C\subset Y$ is can be chosen generically so to that:
(a) the restricted morphism $a_{|C}:C\to A$ is birational;
(b) the restricted co-differential $\Omega^1_{A,0}\otimes\mathcal{O}_C\to\Omega^1_{X\,|C}$ is generically surjective;
(c) the sequence $0\to T_{Y\,|C}\to T_{X\,|C}\to N_{|C}\to 0$ is exact.
Because of (c) the class $e=e^{X,Y}_C$ lies in the vector subspace $H^1(T_{Y\,|C}\otimes N^\vee_{|C})\subset\mathrm{Ext}^1_C(N^{-1}_{|C},\Omega^1_{Y\,|C})$. In the notation introduced in 4.2, we have that $\rho_e=0$, since the co-differential $\Omega_{A,0}\otimes\mathcal{O}_C\to\Omega^1_{Y\,|C}$ lifts to $\Omega^1_{X\,|C}$. Therefore $e\in\ker(\phi^A_n)$. Because of (b) we have that $\Phi^A_n(e)\neq 0$. Therefore hypothesis (ii) of Theorem 4.5 is satisfied. Since hypothesis (i) is satisfied too (because of (a)), Theorem 4.5 can be applied to the tuple $(Y,N,C,\widehat{A},p_Y^*N^n\otimes(a_{|Y},id_{\hat{A}})^*\mathcal{P})$. It follows that $e^{X,Y}_C\in\ker(\phi^{\mathcal{P}_{\hat{A}}}_n)$. Therefore, by Theorem 3.5 the statement holds also for $i=\dim X-1$.

4.7. Theorem *If $a:X\to A$ is a separable morphism from an irreducible, Gorenstein, projective variety to an abelian variety then $R^ip_{A^\vee_*}((a,id_{A^\vee})^*\mathcal{P})=0$ for any $i<\dim a(X)$.*

The Proof of Theorem 4.4. is by induction on $\dim X$ and completely similar to the proof of Theorem 4.6 (take a suitable hyperplane section of dimension equal $\dim a(X)$ such that the restriction $a_{|Z}$ is generically finite and separable and apply Theorem 4.6.) We leave the details to the reader.

5. CO-GAUSSIAN MAPS AND RELATIVE FOURIER-MUKAI FUNCTOR, I

In this section we will show that the global co-gaussian map $\phi^{\mathcal{P}_{\hat{A}}}_n$ of (4.1) has a very natural interpretation in terms of the Fourier-Mukai transform. This will be a key point of the proof of Theorem 4.5.

5.1. Relative Fourier-Mukai functor. Let A be an abelian variety, equipped of a Poincaré line bundle $\mathcal{P}$ on $A \times \widehat{A}$. Given a projective variety Y we consider the abelian scheme $Y \times A$ and its dual scheme $Y \times \widehat{A}$. Via the natural isomorphism $\Phi : (Y \times A) \times_Y (Y \times \widehat{A}) \cong Y \times A \times \widehat{A}$ the line bundle

$$\mathcal{R} = \Phi^*(p_{23}^*\mathcal{P}), \tag{5.1}$$

is a Poincarè line bundle on $(Y \times A) \times_Y (Y \times \widehat{A})$. Denoting π_1 and π_2 the two projections, we have the functor $\widehat{\mathcal{S}} = \pi_{2_*}(\mathcal{R} \otimes \pi_1^* ?)$ (resp. $\mathcal{S} = \pi_{1_*}(\mathcal{R} \otimes \pi_2^* ?)$) taking a coherent sheaf on $Y \times A$ (resp. $Y \times \widehat{A}$)) to a coherent sheaf on $Y \times \widehat{A}$ (resp. $Y \times A$) . The theorem of Mukai ([M2], Theorem 1.1, see also [M1]), states that their derived functors $\mathbf{R}\widehat{\mathcal{S}} : \mathbf{D}(Y \times A) \to \mathbf{D}(Y \times \widehat{A})$ and $\mathbf{R}\mathcal{S} : \mathbf{D}(Y \times \widehat{A}) \to \mathbf{D}(Y \times A)$ are equivalence of categories, as $\mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\mathcal{S} = (-1)^*_{Y \times A}[-g]$ and $\mathbf{R}\mathcal{S} \circ \mathbf{R}\widehat{\mathcal{S}} = (-1)^*_{Y \times \widehat{A}}[-g]$ (where $g = \dim A$). Therefore $\mathbf{R}\widehat{\mathcal{S}}$ is a fully faithful functor. In particular, given coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $Y \times A$, we have the isomorphism

$$\begin{aligned}
\mathrm{Ext}^i_{Y \times A}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathbf{D}(Y \times A)}(\mathcal{F}, \mathcal{G}[i]) \\
&\cong \mathrm{Hom}_{\mathbf{D}(Y \times \widehat{A})}(\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F}), \mathbf{R}\widehat{\mathcal{S}}(\mathcal{G})[i]) \\
&= \mathrm{Ext}^i_{Y \times \widehat{A}}(\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F}), \mathbf{R}\widehat{\mathcal{S}}(\mathcal{G}))
\end{aligned} \tag{5.2}$$

Finally, let us assume that there is a morphism $a : Y \to A$ and let us denote

$$\mathcal{Q} = (a, id_{\widehat{A}})^*\mathcal{P} \tag{5.3}$$

(note that, with respect to the notation used in Set-up 5.1, we are suppressing – for sake of simplicity – the subscript $Y \times \widehat{A}$ from the notation). Let Γ_Y be the graph of the morphism a. An immediate verification shows that

$$\mathbf{R}\widehat{\mathcal{S}}(\mathcal{O}_{\Gamma_Y}) = \widehat{\mathcal{S}}(\mathcal{O}_{\Gamma_Y}) = \mathcal{Q}. \tag{5.4}$$

This generalizes the obvious fact, valid on abelian *varieties*, that $\mathbf{R}\widehat{\mathcal{S}}(k(0)) = \widehat{\mathcal{S}}(k(0)) = \mathcal{O}_{\widehat{A}}$.

5.2. Expressing co-gaussian maps in terms of relative Fourier-Mukai functor. In order to write, as announced, the co-gaussian map ϕ in terms of FM transform, we identify its domain as follows

$$\mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C}) \cong \mathrm{Ext}^1_{Y \times C}(\mathcal{I}_{\Delta^Y_C} \otimes p_C^* N_{|C}, \mathcal{O}_{\Delta^Y_C}) \tag{5.5}$$

where p_Y, p_C and Δ^Y_C denote respectively the two projections of $Y \times C$ and the graph of the embedding $C \overset{i}{\hookrightarrow} Y$. The isomorphism follows by adjunction with respect to the graph-embedding $\delta^Y_C : C \to Y \times C$, $p \mapsto (i(p), p)$ (in fact $\Omega^1_{Y|C} \otimes N_{|C}$ is $\delta^{Y*}_C(\mathcal{I}_{\Delta^Y_C} \otimes p_2^* N_{|C})$, and $\mathcal{O}_{\Delta^Y_C}$ is of course $\delta^Y_{C*}(\mathcal{O}_C)$). Similarly, we have the isomorphism

$$\mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C}) \overset{(1.4)}{\cong} \mathrm{Ext}^n_Y(\Omega^1_{Y|C} \otimes N^n, \mathcal{O}_Y) \cong \mathrm{Ext}^n_{Y \times Y}(\mathcal{I}_{\Delta^Y_C} \otimes p_C^* N^n_{|C}, \mathcal{O}_{\Delta_Y}) \tag{5.6}$$

where Δ_Y is the diagonal of $Y \times Y$. Now the co-gaussian map can be described as follows

5.3. Lemma. $\phi_n^{\mathcal{P}_{\hat{A}}} = \gamma_n \circ FM \circ \alpha_n$ *in the following diagram with exact column:*

$$\begin{array}{ccc} \mathrm{Ext}^1_C(\Omega^1_{Y|C}, N^{-1}_{|C}) & & \mathrm{Ext}^{n+1}_{Y\times\widehat{A}}(R^1\widehat{\mathcal{S}}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_2^*N^n_{|C})), \mathcal{Q}) \\ \downarrow \alpha_n & & \downarrow \delta_n \\ \mathrm{Ext}^n_{Y\times A}((id_Y, a_{|C_*})(\mathcal{I}_{\Delta^Y_C}\otimes p_C^*N^n_{|C}), \mathcal{O}_{\Gamma_Y}) & \overset{FM}{\cong} & \mathrm{Ext}^n_{Y\times\widehat{A}}(\mathbf{R}\widehat{\mathcal{S}}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_2^*N^n_{|C})), \mathcal{Q}) \\ & & \downarrow \gamma_n \\ & & \mathrm{Ext}^n_{Y\times\widehat{A}}(R^0\widehat{\mathcal{S}}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_2^*N^n_{|C})), \mathcal{Q}) \end{array} \tag{5.7}$$

where:

- the horizontal isomorphism is the Fourier-Mukai isomorphism (5.2) (with (5.4) plugged in).

- α_n is most easily seen as the composition of the isomorphism (5.6) with the natural map

$$\mathrm{Ext}^n_{Y\times Y}(\mathcal{I}_{\Delta^Y_C}\otimes p_Y^*N^n_{|C}, \mathcal{O}_{\Delta_Y}) \to \mathrm{Ext}^n_{Y\times A}((id_Y, a_{|Y})_*(\mathcal{I}_{\Delta_Y}\otimes p_Y^*N^n_{|C}), \mathcal{O}_{\Gamma_Y}).$$

The map α_n will be described more thoroughly in 6.4 below.

- the column on the right follows from the hypercohomology spectral sequence computing $\mathrm{Ext}^\bullet_{Y\times A^\vee}(\mathbf{R}\widehat{\mathcal{S}}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_2^*N^n_{|C})), \mathcal{Q})$. Since C is a curve there are only $R^0\widehat{\mathcal{S}}$'s and $R^1\widehat{\mathcal{S}}$'s and the spectral sequence reduces to the long exact sequence

$$\cdots \to \mathrm{Ext}^{p+1}_{Y\times\hat{A}}(R^1\widehat{\mathcal{S}}(?), \mathcal{Q}) \xrightarrow{\delta_p} \mathrm{Ext}^p_{Y\times\hat{A}}(\mathbf{R}\widehat{\mathcal{S}}(?), \mathcal{Q}) \xrightarrow{\gamma_p} \mathrm{Ext}^p_{Y\times\hat{A}}(R^0\widehat{\mathcal{S}}(?), \mathcal{Q}) \to \cdots \tag{5.8}$$

The case $p = n$ applied to $? = (id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_2^*N^n_{|C})$ is the left vertical column of (5.7).

Note that the proof of Lemma 5.3 is immediate since, recalling the definition of $\mathcal{R}$ and $\mathcal{Q}$, we have that $(id_Y, a_{|C}, id_{A^\vee})^*\mathcal{R} \cong p_2^*(\mathcal{Q}_{|C\times\hat{A}})$ (where p_2 denotes, as in §1, the second projection of $(Y\times\widehat{A})\times_{\widehat{A}}(C\times\widehat{A})$). Then the statement follows directly by checking that – keeping track of the various identifications – the maps $\phi_n^{\mathcal{P}_{\hat{A}}}$ and $\gamma_n\circ FM\circ\alpha_n$ are defined in the same way.

In view of Lemma 5.3, Theorem 4.2 is equivalent to the following statement:

5.4. Lemma. (First reduction of Theorem 4.5) *The thesis of Theorem 4.5 is equivalent to the fact that* $FM\big(\alpha_n(\ker(\phi_n^A))\big) \subset \mathrm{Im}(\delta_n)$.

6. CO-GAUSSIAN MAPS AND RELATIVE FOURIER-MUKAI FUNCTOR, II

6.1. Computations on the $\mathbf{Y\times A}$-side of diagram (5.7). Diagram (5.7) relates, via the FM transform, cohomology groups on $Y\times A$ (left side) and on $Y\times\widehat{A}$ (right side). In this section we perform separately some computations on both sides in order to clarify the equivalent formulation supplied by Lemma 5.4. The result (Lemma 6.7 below) will be that it is sufficient to deal with the FM transform of $\mathrm{Ext}^n_{Y\times A}((id_Y, a_{|C})_*(p_C^*N^n_{|C}), \mathcal{O}_{\Gamma_Y}))$ rather than that of $\mathrm{Ext}^n_{Y\times A}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_C^*N^n_{|C}), \mathcal{O}_{\Gamma_Y}))$. We start by the $Y\times A$-side. It is convenient to express the map α_n of diagram (5.7) in terms of the Grothendieck duality spectral sequence

$$\mathrm{Ext}^p_{Y\times C}(?, \mathcal{E}xt^q_{Y\times A}((id_Y\times a_{|C})_*(\mathcal{O}_{Y\times C}), \mathcal{O}_{\Gamma_Y})) \Rightarrow \mathrm{Ext}^{p+q}_{Y\times A}((id_Y\times a_{|C})_*(?), \mathcal{O}_{\Gamma_Y}) \tag{6.1}$$

where, by abuse of language, $\mathcal{E}xt^q_{Y\times A}((id_Y \times a_{|C})_*(\mathcal{O}_{Y\times C}), \mathcal{O}_{\Gamma_Y})$ is seen as a $\mathcal{O}_{Y\times C}$-module via its structure of $(id_Y \times a_{|C})_*(\mathcal{O}_{Y\times C})$-module. Therefore one needs to describe the sheaves $\mathcal{E}xt^q_{Y\times A}((id_Y \times a_{|C})_*(\mathcal{O}_{Y\times C}), \mathcal{O}_{\Gamma_Y})$, at least some of them. This the content of Lemma 6.3 below.

6.2. Definition. (Restricted equisingular normal sheaf) We denote $\mathcal{N}'_a$ the cokernel of the restricted differential $d = (d_{a_Y})_{|C} : T_{Y\,|C} \to T_{A.0} \otimes \mathcal{O}_C$:

$$0 \to T_{Y\,|C} \overset{d}{\to} T_{A.0} \otimes \mathcal{O}_C \to \mathcal{N}'_a \to 0 \tag{6.2}$$

Clearly, we have that $\quad \ker(\phi^A_n) = H^0(\mathcal{N}'_a \otimes N^{-1}_{|C})$.

6.3. Lemma. *Denoting, as above, $\delta^Y_C : C \to \Delta^Y_C \subset Y \times C$ the graph-embedding* (see paragraph 5.2)*, we have that all $\mathcal{E}xt^i_{Y\times A}((id_Y, a_{|C})_*\mathcal{O}_{Y\times C}, \mathcal{O}_{\Gamma_Y})$ are supported on Δ^Y_C. Moreover we have the canonical isomorphisms of $\mathcal{O}_{Y\times A}$-modules:*

$$\mathcal{E}xt^i_{Y\times A}((id_Y, a_{|C})_*\mathcal{O}_{Y\times C}, \mathcal{O}_{\Gamma_Y}) = \begin{cases} 0, & \text{for } i < n-1 \\ \delta^Y_{C\,*} N^{n-1}_{|C}, & \text{for } i = n-1 \\ \delta^Y_{C\,*}\mathcal{N}'_a \otimes N^{n-1}_{|C}, & \text{for } i = n \\ \delta^Y_{C\,*}\omega_C & \text{for } i = g-1 \end{cases}$$

6.4. We postpone the proof of Lemma 6.3 to the end of the section. Now we apply it to diagram (5.7): the map α_n is identified to the edge-map of the spectral sequence (6.1) applied to $\mathcal{I}_{\Delta^Y_C} \otimes p^*_C N^n_{|C}$:

$$\begin{array}{ccc} \mathrm{Ext}^1_C(\Omega^1_{Y\,|C}, N^{-1}_{|C}) & \overset{(5.5)}{\cong} & \mathrm{Ext}^1_{Y\times C}(\mathcal{I}_{\Delta^Y_C} \otimes p^*_C N^n_{|C}, \delta^Y_{C\,*} N^{n-1}_{|C}) \\ & & \downarrow \alpha_n \\ & & \mathrm{Ext}^n_{Y\times A}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C} \otimes p^*_Y N^n_{|C}), \mathcal{O}_{\Gamma_Y}) \end{array} \tag{6.3}$$

(here we are using the first and second assertion of Lemma 6.3). Moreover in the beginning the same spectral sequence applied to $p^*_C N^n_{|C}$ we have the edge map

$$\mathrm{Ext}^n_{Y\times C}(p^*_C N^n_{|C}, \mathcal{O}_{\Gamma_Y}) \overset{\beta_n}{\to} \mathrm{Hom}(p^*_C N^n_{|C}, \delta^Y_{C_*}(\mathcal{N}'_a \otimes N^{n-1}_{|C})) \cong H^0(\mathcal{N}'_a \otimes N^{-1}_{|C}) \tag{6.4}$$

(here we are using the first and third assertion of Lemma 6.3). The maps (6.3) and (6.4) fit together as follows:

6.5. Lemma. *Applying the spectral sequence* (6.1) *to the arrow $\mathcal{I}_{\Delta^Y_C} \otimes p^*_C N^n_{|C} \to p^*_C N^n_{|C}$ we get the commutative diagram*

$$\begin{array}{ccc} H^0(\mathcal{N}'_a \otimes N^{-1}_{|C}) & \to & H^1(T_{Y\,|C} \otimes N^{-1}_{|C}) \subset \mathrm{Ext}^1_C(\Omega^1_{Y\,|C}, N^{-1}_{|C}) \\ \uparrow \beta_n & & \downarrow \alpha_n \\ \mathrm{Ext}^n_{Y\times A}((id_Y, a_{|C})_*(p^*_C N^n_{|C}), \mathcal{O}_{\Gamma_Y}) & \to & \mathrm{Ext}^n_{Y\times A}((id_Y, a_{|C})_*(\mathcal{I}_{\Delta^Y_C} \otimes p^*_C N^n_{|C}), \mathcal{O}_{\Gamma_Y}) \end{array}$$

where the top horizontal map is the coboundary map of sequence (6.2) *tensored with $N^{-1}_{|C}$.*

6.6. Second reduction of Theorem 4.5. Now we come to the main result of the section. For each p have the commutative diagram with exact column

$$
\begin{array}{ccc}
\mathrm{Ext}^{p+1}_{Y\times\widehat{A}}(R^1\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q}) & & \mathrm{Hom}((id_Y,a_{|C})_*(p_C^*N^n_{|C}),\mathcal{E}xt^p_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Gamma_Y})) \\
\downarrow \delta_p & & \uparrow \beta_p \\
\mathrm{Ext}^{p}_{Y\times\widehat{A}}(\mathbf{R}\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q}) & \overset{FM^{-1}}{\cong} & \mathrm{Ext}^p_{Y\times A}((id_Y,a_{|C})_*(p_C^*N^n_{|C}),\mathcal{O}_{\Gamma_Y}) \\
\downarrow \gamma_p & & \\
\mathrm{Ext}^{p}_{Y\times\widehat{A}}(R^0\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q}) & &
\end{array}
\tag{6.5}
$$

where the column is the degenerate spectral sequence (5.10) applied to $((id_Y,a_{|C})_*(p_C^*N^n_{|C}))$ and the map β_p are the edge-maps of the spectral sequence (6.1) applied to $p_C^*N^n_{|C}$. For $p=n$, such diagram is identified, via (6.4)

$$
\begin{array}{ccc}
\mathrm{Ext}^{n+1}_{Y\times\widehat{A}}(R^1\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q}) & & H^0(\mathcal{N}'_a\otimes N^{-1}_{|C}) \\
\downarrow \delta_n & & \uparrow \beta_n \\
\mathrm{Ext}^{n}_{Y\times\widehat{A}}(\mathbf{R}\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q}) & \overset{FM^{-1}}{\cong} & \mathrm{Ext}^n_{Y\times A}((id_Y,a_{|C})_*(p_C^*N^n_{|C}),\mathcal{O}_{\Gamma_Y}) \\
\downarrow \gamma_n & & \\
\mathrm{Ext}^{n}_{Y\times\widehat{A}}(R^0\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q}) & &
\end{array}
\tag{6.6}
$$

6.7 Lemma. (Second reduction of Theorem 4.5) *Theorem 4.5 is equivalent to the fact that in diagram* (6.6) *the composition $\beta_n\circ FM^{-1}\circ\delta_n$ is surjective.*

Proof. We apply $\widehat{\mathcal{S}}\circ(id_Y,a_{|C})_*$ to the exact sequence $0\to\mathcal{I}_{\Delta^Y_C}\otimes p_Y^*N^n_{|C}\to p_Y^*N^n_{|C}\to\delta^Y_{C_*}(N^n_{|C})\to 0$. As we are assuming that $p_Y^*N^n\otimes\mathcal{Q}$ – and a fortiori $p_Y^*N^n_{|C}\otimes\mathcal{Q}$ – is relatively base point-free, we have the isomorphism

$$R^1\widehat{\mathcal{S}}((id_Y,a_{|C})_*(\mathcal{I}_{\Delta^Y_C}\otimes p_C^*N^n_{|C}))\xrightarrow{\sim}R^1\widehat{\mathcal{S}}((id_Y,a_{|C})_*(p_C^*N^n_{|C})),\mathcal{Q})\tag{6.7}$$

The statement follows passing trough the reduction given by Lemma 5.4: the $Y\times\widehat{A}$ part of diagram (5.7) is replaced by the $Y\times\widehat{A}$ part of diagram (6.6) thanks to isomorphism (6.7) while the change in the $Y\times A$-side is allowed by Lemma 6.5.

6.8. Proof of Lemma 6.3. We start with a preliminary Lemma.

6.9 Lemma. *Via the identification $C\cong\Delta^Y_C$*

$$\mathcal{E}xt^q_{Y\times A}((id,a_{|C})_*\mathcal{O}_{\Delta^Y_C},\mathcal{O}_{\Gamma_Y})\cong\Lambda^{q-n+1}T_{A,0}\otimes N^{-n+1}_{|C}.$$

Proof. Since the difference map $d_Y:Y\times A\to A$ defined by $(y,x)\mapsto a(y)-x$ is flat, we have that

$$
\begin{aligned}
\mathcal{E}xt^p_{Y\times A}(\mathcal{O}_{\Gamma_Y},\mathcal{O}_{\Gamma_Y}) &\cong \mathcal{E}xt^p_A(d_Y^*\mathcal{O}_0,d_Y^*\mathcal{O}_0)\\
&\cong d_Y^*\mathcal{E}xt^p_A(\mathcal{O}_0,\mathcal{O}_0)\\
&\cong d_Y^*(\Lambda^pT_{A,0}\otimes\mathcal{O}_0)\\
&\cong \Lambda^pT_{A,0}\otimes\mathcal{O}_{\Gamma_Y}
\end{aligned}\quad .
\tag{6.8}
$$

Moreover we have that

$$\mathcal{E}xt^i_Y(\mathcal{O}_C,\mathcal{O}_Y)\cong\begin{cases} N^{-n+1}_{|C} & \text{for } i=n-1\\ 0 & \text{otherwise}\end{cases} \tag{6.9}$$

Therefore, Lemma 6.9 follows by degeneration of the change-of-ring spectral sequence

$$\mathcal{E}xt^i_{\Gamma_Y}(?,\mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{\Gamma_Y},\mathcal{O}_{\Gamma_Y})\Rightarrow\mathcal{E}xt^{i+j}_{Y\times A}(?,\mathcal{O}_{\Gamma_Y}) \tag{6.10}$$

applied to the $\mathcal{O}_{\Gamma_Y}$-module $\mathcal{O}_{\Gamma_C}$ (we recall that $\Gamma_C\subset C\times A$ the graph of the map $a_{|C}:C\to A$). In fact we get

$$\begin{aligned}\Lambda^{q-n+1}T_{A,0}\otimes N^{-n+1}_{|C} &\overset{(6.9)}{\cong} \mathcal{E}xt^{n-1}_{\Gamma_Y}(\mathcal{O}_{\Gamma_C},\Lambda^{q-n+1}T_{A,0}\otimes\mathcal{O}_{\Gamma_Y})\\ &\overset{(6.8)}{\cong} \mathcal{E}xt^{n-1}_{\Gamma_Y}(\mathcal{O}_{\Gamma_C},\mathcal{E}xt^{q-n+1}_{Y\times A}(\mathcal{O}_{\Gamma_Y},\mathcal{O}_{\Gamma_Y}))\\ &\overset{(6.10)}{\cong} \mathcal{E}xt^{q}_{Y\times A}(\mathcal{O}_{\Gamma_C},\mathcal{O}_{\Gamma_Y})\\ &= \mathcal{E}xt^{q}_{Y\times A}((id,a_{|C})_*\mathcal{O}_{\Delta^Y_C},\mathcal{O}_{\Gamma_Y})\end{aligned}$$

End of Proof of Lemma 6.3. We apply the left-exact functor $\mathcal{H}om_{Y\times A}((id,a_{|C})_*(?),\mathcal{O}_{\Gamma_Y})$ to the exact sequence

$$0\to\mathcal{I}_{\Delta^Y_C}\to\mathcal{O}_{Y\times C}\to\mathcal{O}_{\Delta^Y_C}\to 0 \tag{6.11}$$

and we analyze the resulting long exact sequence. Since $\Gamma_Y\cap(Y\times a(C))=\Gamma_C\cong\Delta^Y_C$ the maps of local $\mathcal{E}xt$'s

$$\mathcal{E}xt^p_{Y\times A}((id,a_{|C})_*\mathcal{O}_{Y\times C},\mathcal{O}_{\Gamma_Y})\to\mathcal{E}xt^p_{Y\times A}((id,a_{|C})_*\mathcal{I}_{\Delta^Y_C},\mathcal{O}_{\Gamma_Y}) \tag{6.12}$$

are zero. Therefore the long exact sequence of $\mathcal{H}om_{Y\times A}((id,a_{|C})_*(?),\mathcal{O}_{\Gamma_Y})$ applied to (6.11) is chopped in short exact sequences

$$0\to\mathcal{E}xt^{q-1}_{Y\times A}((id,a_{|C})_*\mathcal{I}_{\Delta^Y_C},\mathcal{O}_{\Gamma_Y})\to\overset{q-n+1}{\Lambda}T_{A,0}\otimes N^{n-1}_{|C}\to\mathcal{E}xt^{q}_{Y\times A}((id,a_{|C})_*\mathcal{O}_{Y\times C},\mathcal{O}_{\Gamma_Y})\to 0 \tag{6.13}$$

(here we are using Lemma 6.9 to settle the middle term). This yields that all $\mathcal{E}xt^q$'s are supported on Δ^Y_C. Taking $q\le n-1$, it is easily seen that in sequence (6.13) the sheaf on the left has to be torsion, hence zero. This proves the first and the second isomorphism. Taking $q=n$ in (6.13), the map $0\to\mathcal{E}xt^n_{Y\times A}((id,a_{|C})_*\mathcal{I}_{\Delta^Y_C},\mathcal{O}_{\Gamma_Y})\to T_{A,0}\otimes N^{n-1}_{|C}$ is identified to the co-differential map d (see sequence (6.2)), tensored by $N^{n-1}_{|C}$. Therefore the third isomorphism follows. The last isomorphism is proved in a different way: in the first place one observes that, since the map $a_{|C}:C\to A$ is birational onto its image,

$$\mathcal{E}xt^q_{Y\times A}((id_Y,a_{|C})_*(\mathcal{O}_{Y\times C}),\mathcal{O}_{Y\times A})\cong\begin{cases}(id_Y,a_{|C})_*(p^*_C\omega_C) & \text{if } q=g-1\\ 0 & \text{otherwise}\end{cases} \tag{6.14}$$

Then one uses the natural transformation $\mathbf{R}\mathcal{H}om(?,\mathcal{O}_{Y\times A})\otimes^{\mathbf{L}}\mathcal{O}_{\Gamma_Y}\cong\mathbf{R}\mathcal{H}om(?,\mathcal{O}_{\Gamma_Y})$ applied to $?=(id_Y,a_{|C})_*(\mathcal{O}_{Y\times C})$. Plugging (6.14) into the corresponding spectral sequence, one gets

$$\mathcal{E}xt^{g-1}_{Y\times A}((id_Y,a_{|C})_*\mathcal{O}_{Y\times C},\mathcal{O}_{\Gamma_Y})\cong(id_Y,a_{|C})_*(p^*_C\omega_C)\otimes\mathcal{O}_{\Gamma_Y}\cong\delta^Y_{C*}\omega_C.$$

7. PROOF OF THEOREM 4.5

7.1. Multiplicative structure. To prove Theorem 4.5 we will prove the condition provided by Lemma 6.7. The strategy will consist in seeing diagrams (6.5) as the homogeneous pieces of a diagram of graded modules on the exterior algebra $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$ (see diagram 7.4 below). In fact the left column of diagram (6.5) is naturally the homogeneous piece of an exact sequence of graded modules on the algebra $\mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathcal{Q},\mathcal{Q}) = \mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathcal{O}_{Y\times\widehat{A}},\mathcal{O}_{Y\times\widehat{A}})$, which contains, as a graded direct summand, $\mathrm{Ext}^\bullet_{\widehat{A}}(\mathcal{O}_{\widehat{A}},\mathcal{O}_{\widehat{A}}) = \Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$.

Passing to the right part of diagram (6.5), we have, by Fourier-Mukai transform ((5.2) and (5.4)), the isomorphism of graded algebras

$$\mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathcal{Q},\mathcal{Q}) \overset{FM}{\cong} \mathrm{Ext}^\bullet_{Y\times A}(\mathcal{O}_{\Gamma_Y},\mathcal{O}_{\Gamma_Y}) = \mathrm{Ext}^\bullet_{Y\times A}(d_Y^*\mathcal{O}_0, d_Y^*\mathcal{O}_0). \tag{7.1}$$

where $d: Y\times A\to A$ is the difference map $(y,x)\to a(y)-x$. Via the Fourier-Mukai isomorphism the graded subalgebra $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$ of the left hand side of (7.1) goes isomorphically to the graded subalgebra $\mathrm{Ext}^\bullet_A(\mathcal{O}_0,\mathcal{O}_0) = \Lambda^\bullet T_{A,0}$ of the right-hand side (note that such isomorphism is the natural one, i.e. the one induced by double duality: $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}}) \cong \Lambda^\bullet T_{\widehat{\widehat{A}},0} \cong \Lambda^\bullet T_{A,0}$). In conclusion, $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$ acts naturally on $\mathrm{Ext}^\bullet_{Y\times A}(\mathcal{F},\mathcal{O}_{\Gamma_Y})$ where, for sake of simplicity, we denote

$$\mathcal{F} = (id_Y, a_{|C})_*(p_C^* N^n_{|C}). \tag{7.2}$$

Moreover $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$ acts also on $\mathrm{Hom}(\mathcal{F}, \mathcal{E}xt^\bullet_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Gamma_Y}))$ and on the map $\beta^\bullet$ of (6.5), via its natural action on $\mathcal{E}xt^\bullet_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Gamma_Y})$. In fact $\mathcal{E}xt^\bullet_{Y\times A}(\mathcal{O}_{\Gamma_Y},\mathcal{O}_{\Gamma_Y}) = d^*\mathcal{E}xt^\bullet_A(\mathcal{O}_0,\mathcal{O}_0) \cong \Lambda^\bullet T_{A,0}\otimes\mathcal{O}_{\Gamma_Y}$.

7.2. Notation. (a) We will identify, via Fourier-Mukai transform, the $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$-modules

$$V^\bullet : \mathrm{Ext}^\bullet_{Y\times A}(\mathcal{F},\mathcal{O}_{\Gamma_Y}) \cong \mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F}),\mathcal{Q}) \tag{7.3}$$

so to see diagrams (6.5) as a commutative diagram, with exact row, of $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$-modules

$$\begin{array}{ccccc} A^\bullet & \overset{\delta^\bullet}{\to} & V^\bullet & \overset{\gamma^\bullet}{\to} & B^\bullet \\ & & \downarrow \beta^\bullet & & \\ & & Z^\bullet & & \end{array} \tag{7.4}$$

(b) We denote

$$\Phi_V^{i,j} : V^i \to \mathrm{Hom}_k(\Lambda^{j-i}H^1(\mathcal{O}_{\widehat{A}}), V^j) \tag{7.5}$$

the map induced by the multiplication map. We will denote Φ_A, Φ_B, Φ_Z and Φ_K the analogous maps for the modules $A^\bullet$, $B^\bullet$, $Z^\bullet$ and $K^\bullet := \ker(\beta^\bullet)$.

By Lemma 6.7, Theorem 4.5 means that $\beta^\bullet\circ\delta^\bullet$ is surjective in degree n. We start by proving that $\beta^\bullet\circ\delta^\bullet$ is surjective in degree $g-1$. This is the content of the Lemma below. In order to state it, let us recall that, by the last isomorphism of Lemma 6.3, we have the identification

$$Z^{g-1} \cong \mathrm{Hom}(N^n_{|C},\omega_C) \tag{7.6}$$

7.3. Lemma. *(i) There is a natural inclusion* $i : H^1(C, N^n_{|C})^\vee \hookrightarrow A^{g-1}$ *such that the map* $\beta^{g-1} \circ \delta^{g-1} \circ i$*, is an isomorphism* (in fact, via the identification (7.6), it is the Serre duality isomorphism.)
(ii) The splitting $V^{g-1} \cong Z^{g-1} \oplus \ker(\beta^{g-1})$ *induced by (i) is natural with respect to the multiplicative structure, in the sense that it induces the splitting* $\mathrm{Im}(\Phi^{i,g-1}_V) \cong \mathrm{Im}(\Phi^{i,g-1}_Z) \oplus \mathrm{Im}(\Phi^{i,g-1}_K)$.

Proof. It is here where we use the hypothesis that Y has Gorenstein singularities. In fact, as the dualizing sheaf ω_Y is invertible, recalling the notation (7.2), we have

$$\begin{aligned}\mathrm{Ext}^g_{Y\times\widehat{A}}(R^1\widehat{\mathcal{S}}(\mathcal{F}), \mathcal{Q}) &\cong \mathrm{Ext}^g_{Y\times\widehat{A}}(R^1\widehat{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{Q}^\vee \otimes p_Y^*\omega_Y, p_Y^*\omega_Y)\\ &\cong H^n(Y \times \widehat{A}, R^1\widehat{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{Q}^\vee \otimes p_Y^*\omega_Y)^\vee\end{aligned}$$

where the last isomorphism is Serre-duality. Next, one observes that, by flat base change,

$$R^1\widehat{\mathcal{S}}(\mathcal{F}) = R^1\widehat{\mathcal{S}}((id_Y, a_{|C})_*(p_C^* N^n_{|C})) \cong p^*_{\widehat{A}} R^1 p_{\widehat{A}_*}(p_A^*(a_{|C_*} N^n_{|C}) \otimes \mathcal{P}) \tag{7.7}$$

Therefore the Leray spectral sequence of the projection $p_{\widehat{A}} : Y \times \widehat{A} \to \widehat{A}$ supplies the edge-map

$$\begin{aligned}&H^n(Y \times \widehat{A}, R^1\widehat{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{Q}^\vee \otimes \omega_Y) \xrightarrow{f}\\ &\xrightarrow{f} H^0(\widehat{A}, R^n p_{\widehat{A}_*}(R^1\widehat{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{Q}^\vee \otimes p_Y^*\omega_Y))\\ &\overset{(7.7)}{\cong} H^0(\widehat{A}, R^n p_{\widehat{A}_*}(p^*_{\widehat{A}}(R^1 p_{\widehat{A}_*}(p_A^*(a_{|C_*}N^n_{|C}) \otimes \mathcal{P}) \otimes \mathcal{Q}^\vee \otimes p_Y^*\omega_Y))\\ &\cong H^0(\widehat{A}, R^1 p_{\widehat{A}_*}(p_A^*(a_{|C_*}N^n_{|C}) \otimes \mathcal{P})) \otimes R^n p_{\widehat{A}_*}(\mathcal{Q}^\vee \otimes p_Y^*\omega_Y))\end{aligned}$$

where the last isomorphism is projection formula. Evaluating at $\hat{0} \in \widehat{A}$ we have the map

$$ev_{\hat{0}} : H^0(\widehat{A}, R^1 p_{\widehat{A}_*}(p_A^*(a_{|C_*}N^n_{|C}) \otimes \mathcal{P})) \otimes R^n p_{\widehat{A}_*}(\mathcal{Q}^\vee \otimes p_Y^*\omega_Y)) \to H^1(C, N^n_{|C}) \otimes H^n(Y, \omega_Y).$$

The required map i is the dual of $ev_{\hat{0}} \circ f$ (after the natural identification $H^n(Y, \omega_Y) \cong k$). The last assertion of (i), i.e. that the map $\beta^{g-1} \circ \delta^{g-1} \circ i$ is identified, via (7.6), to Serre duality follows by construction, as well as point (ii), i.e. the compatibility with the multiplicative structure.

At this point we invoke the following Lemma, expressing the compatibility of the multiplicative structure with respect to the spectral sequence (5.8). We use Notation 7.2.

7.4. Lemma. *Let us consider the direct summand* $Z^{g-1} \cong H^1(C, N^n_{|C})^\vee \subset \mathrm{Im}(\delta_{g-1})$ *provided Lemma 7.3. Then* $(\Phi^{n,g-1}_V)^{-1}(Z^{g-1}) \subset \mathrm{Im}(\delta_n) + \ker(\Phi^{n,g-1}_V)$.

7.5. Conclusion of the proof. Granting the Lemma for the time being, let us conclude the proof of the Theorem 4.5. By the splitting supplied by Lemma 7.3(ii) we have that $\mathrm{Im}(\Phi^{n,g-1}_Z) \subset Z^{g-1} \cap \mathrm{Im}(\Phi^{n,g-1}_V)$. By Lemma 7.4, $(\Phi^{n,g-1}_V)^{-1}(\mathrm{Im}(\Phi^{n,g-1}_Z))$ is contained in $\mathrm{Im}(\delta^n) + \ker(\Phi^{n,g-1}_V)$. On the other hand, again by Lemma 7.3(ii), $(\Phi^{n,g-1}_V)^{-1}(\mathrm{Im}(\Phi^{n,g-1}_Z)$ surjects onto Z_n via the surjective map β^n. This proves that β^n, restricted to $\mathrm{Im}(\delta_n) + \ker(\Phi^{n,g-1}_V)$ surjects onto Z_n. Note that the multiplication map $\Phi^{n,g-1}_Z : Z^n \to \mathrm{Hom}_k(\Lambda^{g-n-1}H^1(\mathcal{O}_{A^\vee}), Z^{g-1})$ is the map ϕ^A_n of the statement of Theorem 4.5, which is assumed to be non-zero. This yields that $(\Phi^{n,g-1}_V)^{-1}(\mathrm{Im}(\Phi^{n,g-1}_Z)$ is not

contained in $\ker(\Phi_V^{n,g-1})$. Therefore the image of the map β_n, restricted to $\mathrm{Im}(\delta_n)$, surjects onto the non-empty Zariski open set $Z_n - \ker(\Phi_Z^{n,g-1})$. Therefore it surjects onto Z_n.

7.6. Proof of Lemma 7.4. This should be the particular case of a more general statement. However, since the author could not find a reference, it seems safer include a ad-hoc proof. We start with a preliminary remark: besides the structure of $H^\bullet(\mathcal{O}_{Y\times\widehat{A}})$-module (whence the structure of $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})$-module), the graded vector space $V^\bullet$ has also a structure of $H^\bullet(\mathcal{O}_C)$-module. In fact $V^\bullet$ is naturally a module over $H^\bullet(\mathcal{O}_{Y\times C\times\widehat{A}})\cong H^\bullet(\mathcal{O}_Y)\otimes H^\bullet(\mathcal{O}_C)\otimes H^\bullet(\mathcal{O}_{\widehat{A}})$. This follows from the definition of relative Fourier functor (see Paragraph 5.1) and the fact that in our case the Fourier functor is applied to the sheaf $\mathcal{F}=(id_Y,a_{|C})_*(p_C^*N^n_{|C})$ which is in fact a sheaf on $Y\times a(C)$. Hence

$$\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F})\cong \mathbf{R}p_{Y\times\widehat{A}_*}(p_C^*N^n_{|C}\otimes p^*_{C\times\widehat{A}}((a_{|C},id_{\widehat{A}})^*\mathcal{P})),$$

where now all the projections take place on $Y\times C\times\widehat{A}$. Therefore, by Grothendieck duality,

$$\mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F}),\mathcal{Q})\cong \mathrm{Ext}^\bullet_{Y\times C\times\widehat{A}}(p_C^*N^n_{|C}\otimes p^*_{C\times\widehat{A}}((a_{|C},id_{\widehat{A}})^*\mathcal{P}),p_C^*\omega_C\otimes p^*_{Y\times\widehat{A}}\mathcal{Q})$$

which is a module over $\mathrm{Ext}^\bullet_{Y\times C\times\widehat{A}}(p_C^*\omega_C,p_C^*\omega_C)=H^\bullet(\mathcal{O}_{Y\times C\times\widehat{A}})$. The multiplicative structure on the algebra $H^\bullet(\mathcal{O}_C)$, i.e. the multiplication map

$$\Psi_V^{i,i+1}:V^i\to \mathrm{Hom}_k(H^1(\mathcal{O}_C),V^{i+1}) \tag{7.10}$$

is described by the fact that, in the exact sequence given by the column of diagram (6.5) (i.e. the row of (7.4)), A^i is sent to B^{i+1} and B^i is sent to zero. Therefore

$$\Psi_V^{i,i+1}(V^i)=\psi_V^{i,i+1}(\mathrm{Im}(\delta_i)). \tag{7.11}$$

7.7. Claim. *The restriction of $\Psi_V^{g-1,g}$ to the direct summand Z^{g-1} is injective.*

Proof. The spectral sequence (6.1) goes as

$$\cdots\to V^{g-1}\stackrel{\beta_{g-1}}{\to}\mathrm{Hom}(N,\omega_C)\to \mathrm{Ext}^1(N,\omega_C)\stackrel{\eta_g}{\to}V^g\to\cdots \tag{7.12}$$

Here we are using Notation 7.2(a), Lemma 6.3, as in (7.5), and the fact that there are only Ext^1's and Hom's, since, again by Lemma 6.3, all the $\mathcal{E}xt^i_{Y\times A}((id,a_{|C})_*\mathcal{I}_{\Delta_C^Y},\mathcal{O}_{\Gamma_Y})$'s are supported on Δ_C^Y. By definition, $Z^{g-1}=\mathrm{Hom}(N,\omega_C)$ and it follows easily that the map $\Psi_V^{g-1,g}$, restricted to Z^{g-1}, is the composition of the usual map $m:\mathrm{Hom}(N,\omega_C)\to\mathrm{Hom}_k(H^1(\mathcal{O}_C),\mathrm{Ext}^1(N,\omega_C))$ and $\mathrm{Hom}(H^1(\mathcal{O}_C),\eta_g))$. The map m is injective (its dual is the cup product $H^0(L)\otimes H^1(\mathcal{O}_C)\to H^1(L)$, which is clearly surjective). The map η_g is injective too by the exact complex (7.12) since, by Lemma 7.3, the map β_{g-1} is surjective. Therefore Claim 7.7 follows.

We are now ready to prove Lemma 7.4. We consider the multiplication maps of the structure of $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}})\otimes H^\bullet(\mathcal{O}_C)$-module:

$$\Theta_V^{n,g}:V^n\to\mathrm{Hom}_k\Big(\big(\Lambda^{g-n-1}H^1(\mathcal{O}_{\widehat{A}})\big)\otimes H^1(\mathcal{O}_C),V_g\Big).$$

Let $a\in V^n$ such that $\Phi_V^{n,g-1}(a)$ (see Notation 7.2) is non-zero and lies in Z^{g-1}. Then we apply $\Psi_V^{g-1,g}$ and it follows by Claim 7.7 that $\theta_V^{n,g-1,g}(a)$ is non zero. On the other hand, we can apply

to a the map $\Psi_V^{n,n+1} : V^n \to \mathrm{Hom}_k(H^1(\mathcal{O}_C), V^{n+1})$ first and then the map $\Phi_V^{n+1,g} : V^{n+1} \to \mathrm{Hom}_k(\Lambda^{g-n-1}H^1(\mathcal{O}_{A^\vee}), V^g)$. As we know that the result is non-zero, $\Psi_V^{n,n+1}(a)$ has to be non-zero and this, by (7.11), happens only if $a \in \mathrm{Im}(\delta_n)$.

7.8. Remark. The Gorenstein hypothesis was used only at one technical point, the construction of the linear map i of Lemma 7.3. It would be interesting to know if such hypothesis can be weakened.

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